

Why the Jacobian Transforms Variables in an Integral—C.E. Mungan, Spring 2011

A quick sketch is presented here of why the Jacobian is needed to transform variables in an integral. I specifically consider the idea of changing two independent variables from $f(x,y)$ to $f(u,v)$ but the same concepts hold for transforming more than two variables.

Suppose we make a differential displacement from some initial point to some final point in the xy -plane,

$$d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} = (1\hat{\mathbf{i}} + 0\hat{\mathbf{j}})dx + (0\hat{\mathbf{i}} + 1\hat{\mathbf{j}})dy. \quad (1)$$

The area element in the original coordinates can be written as

$$dA = dxdy = \begin{vmatrix} dx & 0 \\ 0 & dy \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} dxdy. \quad (2)$$

Note that the elements of the first row of the second determinant are simply the components of the vector multiplying dx , and similarly the elements of the second row are the components of the vector multiplying dy in the rightmost expression in Eq. (1). This idea is just the familiar parallelogram rule for evaluating areas as the magnitude of the cross-product of vectors spanning their edges.¹

Next recall that Taylor's theorem enables us to write

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \quad \text{and} \quad dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv. \quad (3)$$

Substitute these two relations into the middle expression in Eq. (1) and rearrange it to get

$$d\mathbf{r} = \left(\frac{\partial x}{\partial u} \hat{\mathbf{i}} + \frac{\partial y}{\partial u} \hat{\mathbf{j}} \right) du + \left(\frac{\partial x}{\partial v} \hat{\mathbf{i}} + \frac{\partial y}{\partial v} \hat{\mathbf{j}} \right) dv. \quad (4)$$

Using the same logic presented after Eq. (2), we can now write the area element in the new coordinates as²

$$dA' = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} dudv = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv \equiv \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \quad (5)$$

where the final determinant defines the Jacobian. (Note that one can either systematically put ∂x terms in the numerators of the first row or ∂u terms in the denominators of the first row to get the Jacobian, so you do not have to remember which to do, as long as you are consistent.)

Finally, the integral now transforms as

$$\int f(x,y)dA = \int f(u,v)dA' \Rightarrow \iint f(x,y)dxdy = \iint f(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \quad (6)$$

as we wanted to show. As a concrete example, consider the transformation from rectangular to spherical coordinates,

$$\begin{aligned}x &= r \sin \theta \cos \phi \\y &= r \sin \theta \sin \phi \\z &= r \cos \theta\end{aligned}\tag{7}$$

so that³

$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} \right| = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix} = r^2 \sin \theta ,\tag{8}$$

i.e., $dV' = dr r d\theta r \sin \theta d\phi$, correctly reproducing the usual formula for the volume element in spherical coordinates.

Endnotes:

1. To extend the idea to 3D, recall that the volume of a parallelepiped whose edges are spanned by vectors \mathbf{A} , \mathbf{B} , and \mathbf{C} is given the triple scalar product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ which can be evaluated as

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}\tag{9}$$

assuming the vectors are in right-hand order.

2. Physicists normally would not put a prime on the transformed area (or volume) element, even though dA and dA' are only conceptually and not mathematically equal. On the other hand, mathematicians would insist that $f'(u, v)$ also ought to be primed. As a specific example, consider transforming the function $f(x, y) = x^2 + y^2$ into plane polar coordinates. A physicist would say the answer is $f(r, \phi) = r^2$, while a mathematician would say that's $f'(r, \phi)$ and that $f(r, \phi) = r^2 + \phi^2$ because "x" and "y" are just slots into which any quantity or variable of interest can be placed. My opinion is one should not lose sleep over this notational issue, but one should be aware of it.

3. The easiest way to evaluate this determinant is to expand in cofactors along the third row.