A frog wants to jump over a cylindrical log lying on flat ground. The log has cross-sectional radius $r$. What is the minimum takeoff speed required to clear the log? The frog can jump from any point on the ground it chooses.

The frog will cross a distance $h$ above the topmost point of the log and will be traveling purely horizontally at that point with speed $v_i$. It will then tangentially brush the log at a point with coordinates $(x, y)$ relative to the center of the log. Suppose that point is at angle $\theta$ relative to the vertical and that the frog is then traveling with speed $v$. Finally it hits the ground with speed $v_f$ at angle $\theta_f$ relative to the ground. (By symmetry these are also the launch speed and angle, and the frog also brushes the log on the way up at angle $-\theta$.)

I will solve the problem in two steps. First, given $h$ I will find the unique value of $v_i$ that causes the frog to brush the log at one point, and from that calculate the landing speed $v_f$. Then I will vary $h$ until this landing speed is minimized.

The four equations of kinematics for the position and velocity of the frog at the brush point are

\[ x = v_i t \quad \Rightarrow \quad t = \frac{x}{v_i} \quad (1) \]

for the flight time from the top point of the frog’s parabolic trajectory, and

\[ r + h - y = \frac{1}{2} g t^2 = \frac{g x^2}{2 v_i^2} \quad (2) \]

using Eq. (1), and

\[ v_x = v_i \quad (3) \]
and finally
\[ v_y = gt = \frac{gx}{v_i} \quad (4) \]

using Eq. (1) again. But since the frog is moving tangentially to the log’s surface as it brushes by,
\[ \tan \theta = \frac{v_y}{v_x} = \frac{x}{y} \quad \Rightarrow \quad y = \frac{v_i^2}{g} \quad (5) \]

using Eqs. (3) and (4). Since the log is circular in cross section, one sees that
\[ x^2 + y^2 = r^2 \quad \Rightarrow \quad x^2 = r^2 - \frac{v_i^4}{g^2} \quad (6) \]

using Eq. (5). We substitute Eqs. (5) and (6) into (2) and rearrange to obtain
\[ \frac{v_i^4}{2g} - (r + h)v_i^2 + \frac{g r^2}{2} = 0. \quad (7) \]

The smaller of the two positive roots of this quadratic equation is
\[ V_i^2 = 1 + H - \sqrt{H^2 + 2H} \quad (8) \]

where I have introduced the dimensionless variables \( H \equiv h/r \) and \( V \equiv v / \sqrt{gr} \). [The larger root, with a plus sign in place of the minus sign in Eq. (8), is unphysical because Eq. (5) would then imply that \( y > r \).] Now using conservation of energy, one finds
\[ v_i^2 = v_i^2 + 2g(2r + h) \quad \Rightarrow \quad V_i^2 = 5 + 3H - \sqrt{H^2 + 2H}. \quad (9) \]

Taking the derivative of this with respect to \( H \) and setting the result to zero gives
\[ 8H^2 + 16H - 1 = 0. \quad (10) \]

The positive root of this quadratic equation is
\[ H = \frac{3}{4} \sqrt{2} - 1 \equiv 0.061. \quad (11) \]

That is, the frog passes above the log by a distance equal to 6% of its radius. Inserting this result into Eq. (9) now gives
\[ V_i^2 = 2 + 2\sqrt{2} \equiv 4.83. \quad (12) \]

Satisfyingly, this is between 4 and 5, the former being the speed of the frog if it dropped from rest from a height equal to the log, and the latter being the landing speed of the frog if it brushed the top of the log so that \( H = 0 \) in Eq. (9).

Equation (12) is the requested solution, but there are a few other variables worth calculating. Substituting Eq. (11) into (8) gives
\[ V_i^2 = \frac{1}{\sqrt{2}} \equiv 0.71. \]  

(13)

Now from Eqs. (5) and (6),
\[
\tan \theta = \frac{x}{V_i^2 r} = \sqrt{V_i^4 - 1} \Rightarrow \theta = 45^\circ
\]

(14)

which is a pleasingly simple result. Finally, the launch angle is exactly
\[
\tan \theta_f = \frac{v_{fy}}{v_{fx}} = \sqrt{V_f^2 - V_i^2} = \sqrt{\left( \frac{V_f}{V_i} \right)^2 - 1} \Rightarrow \theta_f = 67.5^\circ.
\]

(15)

The frog lands a horizontal distance from the center of the log given by Eq. (2) with \( y = -r \),
\[
X_f^2 = (2 + H)2V_i^2 = \left( 1 + 1/\sqrt{2} \right)^2 \Rightarrow X_f \equiv 1.71
\]

(16)

where \( X \equiv x/r \). One can now compute the trajectory \( Y(X) \equiv y/r \) from Eq. (2). Here is an Excel plot of the result.