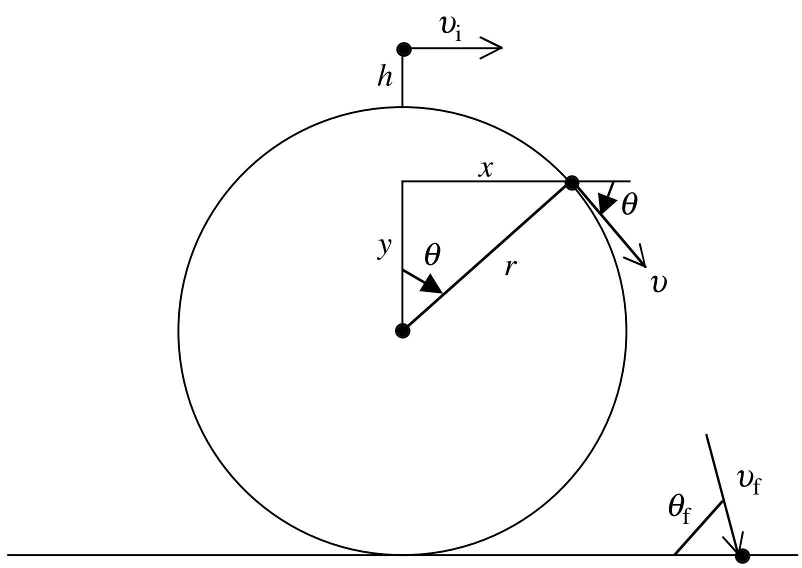
The Jumping Frog—C.E. Mungan, Fall 2004

A frog wants to jump over a cylindrical log lying on flat ground. The log has cross-sectional radius r. What is the minimum takeoff speed required to clear the log? The frog can jump from any point on the ground it chooses.

The frog will cross a distance h above the topmost point of the log and will be traveling purely horizontally at that point with speed υ_i. It will then tangentially brush the log at a point with coordinates (x, y) relative to the center of the log. Suppose that point is at angle θ relative to the vertical and that the frog is then traveling with speed υ. Finally it hits the ground with speed υ_f at angle θ_f relative to the ground. (By symmetry these are also the launch speed and angle, and the frog also brushes the log on the way up at angle $-\theta$.)



I will solve the problem in two steps. First, given h I will find the unique value of υ_i that causes the frog to brush the log at one point, and from that calculate the landing speed υ_f. Then I will vary h until this landing speed is minimized.

The four equations of kinematics for the position and velocity of the frog at the brush point are

$$x = \upsilon_i t \quad \Rightarrow \quad t = \frac{x}{\upsilon_i} \tag{1}$$

for the flight time from the top point of the frog's parabolic trajectory, and

$$r + h - y = \frac{1}{2} g t^2 = \frac{g x^2}{2 \upsilon_i^2} \tag{2}$$

using Eq. (1), and

$$\upsilon_x = \upsilon_i \tag{3}$$

and finally

$$\upsilon_y = gt = \frac{gx}{\upsilon_\mathrm{i}} \tag{4}$$

using Eq. (1) again. But since the frog is moving tangentially to the log's surface as it brushes by,

$$\tan\theta = \frac{\upsilon_y}{\upsilon_x} = \frac{x}{y} \quad \Rightarrow \quad y = \frac{\upsilon_\mathrm{i}^2}{g} \tag{5}$$

using Eqs. (3) and (4). Since the log is circular in cross section, one sees that

$$x^2 + y^2 = r^2 \quad \Rightarrow \quad x^2 = r^2 - \frac{\upsilon_\mathrm{i}^4}{g^2} \tag{6}$$

using Eq. (5). We substitute Eqs. (5) and (6) into (2) and rearrange to obtain

$$\frac{\upsilon_\mathrm{i}^4}{2g} - (r+h)\upsilon_\mathrm{i}^2 + \frac{gr^2}{2} = 0\,. \tag{7}$$

The smaller of the two positive roots of this quadratic equation is

$$V_\mathrm{i}^2 = 1 + H - \sqrt{H^2 + 2H} \tag{8}$$

where I have introduced the dimensionless variables $H \equiv h/r$ and $V \equiv \upsilon/\sqrt{gr}$. [The larger root, with a plus sign in place of the minus sign in Eq. (8), is unphysical because Eq. (5) would then imply that $y > r$.] Now using conservation of energy, one finds

$$\upsilon_\mathrm{f}^2 = \upsilon_\mathrm{i}^2 + 2g(2r+h) \quad \Rightarrow \quad V_\mathrm{f}^2 = 5 + 3H - \sqrt{H^2 + 2H}\,. \tag{9}$$

Taking the derivative of this with respect to H and setting the result to zero gives

$$8H^2 + 16H - 1 = 0\,. \tag{10}$$

The positive root of this quadratic equation is

$$H = \tfrac{3}{4}\sqrt{2} - 1 \cong 0.061\,. \tag{11}$$

That is, the frog passes above the log by a distance equal to 6% of its radius. Inserting this result into Eq. (9) now gives

$$V_\mathrm{f}^2 = 2 + 2\sqrt{2} \cong 4.83\,. \tag{12}$$

Satisfyingly, this is between 4 and 5, the former being the speed of the frog if it dropped from rest from a height equal to the log, and the latter being the landing speed of the frog if it brushed the top of the log so that $H = 0$ in Eq. (9).

Equation (12) is the requested solution, but there are a few other variables worth calculating. Substituting Eq. (11) into (8) gives

$$V_i^2 = \frac{1}{\sqrt{2}} \cong 0.71\,. \tag{13}$$

Now from Eqs. (5) and (6),

$$\tan\theta = \frac{x}{V_i^2 r} = \sqrt{V_i^{-4} - 1} \quad \Rightarrow \quad \theta = 45^\circ \tag{14}$$

which is a pleasingly simple result. Finally, the launch angle is exactly

$$\tan\theta_f = \frac{\upsilon_{fy}}{\upsilon_{fx}} = \frac{\sqrt{\upsilon_f^2 - \upsilon_i^2}}{\upsilon_i} = \sqrt{\left(\frac{V_f}{V_i}\right)^2 - 1} \quad \Rightarrow \quad \theta_f = 67.5^\circ\,. \tag{15}$$

The frog lands a horizontal distance from the center of the log given by Eq. (2) with $y = -r$,

$$X_f^2 = (2+H)2V_i^2 = \left(1 + 1/\sqrt{2}\right)^2 \quad \Rightarrow \quad X_f \cong 1.71 \tag{16}$$

where $X \equiv x/r$. One can now compute the trajectory $Y(X) \equiv y/r$ from Eq. (2). Here is an Excel plot of the result.