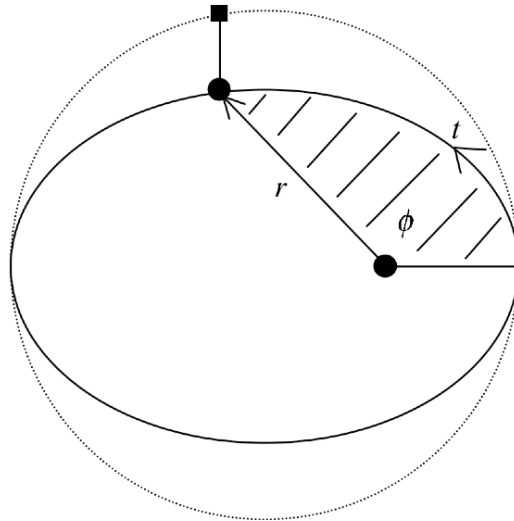


## Kepler's Equation—C.E. Mungan, Fall 2004

A satellite is orbiting a body on an ellipse. Denote the position of the satellite relative to the body by plane polar coordinates  $(r, \phi)$ . Find the time  $t$  that it requires the satellite to move from periapsis (position of closest approach) to angular position  $\phi$ . Express your answer in terms of the eccentricity  $e$  of the orbit and the period  $T$  for a full revolution.



According to Kepler's second law, the ratio of the shaded area  $A$  to the total area  $\pi ab$  of the ellipse (with semi-major axis  $a$  and semi-minor axis  $b$ ) is the respective ratio of transit times,

$$\frac{A}{\pi ab} = \frac{t}{T}. \quad (1)$$

But we can divide the shaded area into differential triangles, of base  $r$  and height  $r d\phi$ , so that

$$A = \frac{1}{2} \int_0^\phi r^2 d\phi \quad (2)$$

where the polar equation of a Keplerian orbit is

$$r = \frac{c}{1 + e \cos \phi} \quad (3)$$

with  $c$  the semilatus rectum (distance vertically from the origin on the diagram above to the ellipse). It is not hard to show from the geometrical properties of an ellipse that

$$b = a\sqrt{1 - e^2} \quad \text{and} \quad c = a(1 - e^2). \quad (4)$$

Substituting (3) into (2), and then (2) and (4) into (1) gives

$$t = \frac{T}{2\pi} M \quad \text{where} \quad M \equiv \int_0^\phi \frac{(1 - e^2)^{3/2}}{(1 + e \cos \phi)^2} d\phi. \quad (5)$$

In the literature,  $M$  is called the mean anomaly. This is an integral solution to the problem. It turns out however that this integral can be evaluated analytically using the following clever change of variables,

$$\cos E = \frac{e + \cos \phi}{1 + e \cos \phi}. \quad (6)$$

Here  $E$  is called the eccentric anomaly and its geometric significance is as follows. Consider the dotted circle of radius  $a$  circumscribing the ellipse in the diagram at the top of the preceding page. As indicated, draw a vertical line upward from the position of the satellite until it intersects the circle at the solid square. Then  $E$  is the angular position of that square about the center of the dotted circle (relative to periapsis). To prove this, draw a line of length  $a$  from the center of the dotted circle to the square. Then its horizontal component is

$$a \cos E = ae + r \cos \phi \quad (7)$$

since the focal length of the ellipse (distance from the geometric center to the focus of the ellipse) is  $ae$ . (Keep in mind that, as drawn, both  $\cos \phi$  and  $\cos E$  are negative. You may wish to redraw the satellite located at an acute angle.) Now substitute (3) and the second equality in (4) into (7) to get (6).

Anyhow, use the trig identity  $\sin^2 E = 1 - \cos^2 E$  to convert (6) into

$$\sin E = \frac{\sin \phi}{1 + e \cos \phi} \sqrt{1 - e^2}. \quad (8)$$

Using this, it is straightforward to differentiate Eq. (6) and prove that

$$\frac{dE}{d\phi} = \frac{\sin E}{\sin \phi} \Rightarrow dE = \frac{\sqrt{1 - e^2}}{1 + e \cos \phi} d\phi. \quad (9)$$

Furthermore, one can easily solve (6) for  $\cos \phi$  to establish that

$$\cos \phi = \frac{\cos E - e}{1 - e \cos E} \Rightarrow 1 - e \cos E = \frac{1 - e^2}{1 + e \cos \phi}. \quad (10)$$

The product of Eqs. (9) and (10) now miraculously gives

$$M = \int_0^E (1 - e \cos E) dE = E - e \sin E, \quad (11)$$

a result known as Kepler's equation. (Kepler managed to deduce this equation geometrically, since calculus had of course not yet been invented by Newton!)

Putting the pieces together, we now have the desired final result,

$$t = \frac{T}{2\pi} \left[ \cos^{-1} \left( \frac{e + \cos \phi}{1 + e \cos \phi} \right) - e \sqrt{1 - e^2} \frac{\sin \phi}{1 + e \cos \phi} \right]. \quad (12)$$

Here are a couple of simple cases that check out. If the satellite's orbit is circular, then  $e = 0$  and we obtain  $t / T = \phi / 2\pi$  as expected. As a second check, at apoapsis we have  $\phi = \pi$  so that  $t = T / 2$ , again as expected.

In practice, one often wishes to know the inverse function: At some specified time  $t$ , at what angular position  $\phi$  will the satellite be located? Given  $t$  we can compute  $M$  from the first equality in (5). Suppose we could then solve Eq. (11) for  $E$ . Then we could put this into the first equality in (10) to get the desired answer  $\phi$ . Unfortunately, Eq. (11) is a transcendental function of  $E$  and cannot be inverted analytically. One needs to solve it either graphically or by iteration from a starting guess. (The latter method works particularly well if  $e$  is small, as is often the case for real orbits.) A numerical solver is available online at <http://www.geocities.com/SiliconValley/2902/kepler.htm>.

Incidentally, now that we have a solution starting from periapsis, it is easy to start and end at arbitrary positions instead. Simply take the difference between Eq. (12) evaluated twice, once with  $\phi = \phi_f$  and then with  $\phi = \phi_i$ .