A Brief Primer on the Lambert *W* Function—C.E. Mungan, Fall 2019

*reference: AJP **87**, 752 (Sep. 2019)*

Suppose we wish to solve the following equation for x,

$$e^{ax} = bx + c \tag{1}$$

given values of a, b, and c. This equation can be rewritten as

$$b(x + c/b)e^{-ax} = 1 \quad \Rightarrow \quad (-ax - ac/b)e^{-ax-ac/b} = -\frac{a}{b}e^{-ac/b}. \tag{2}$$

So if we define two new variables

$$z \equiv -\frac{a}{b}e^{-ac/b} \quad \text{and} \quad W \equiv -ax - ac/b \tag{3}$$

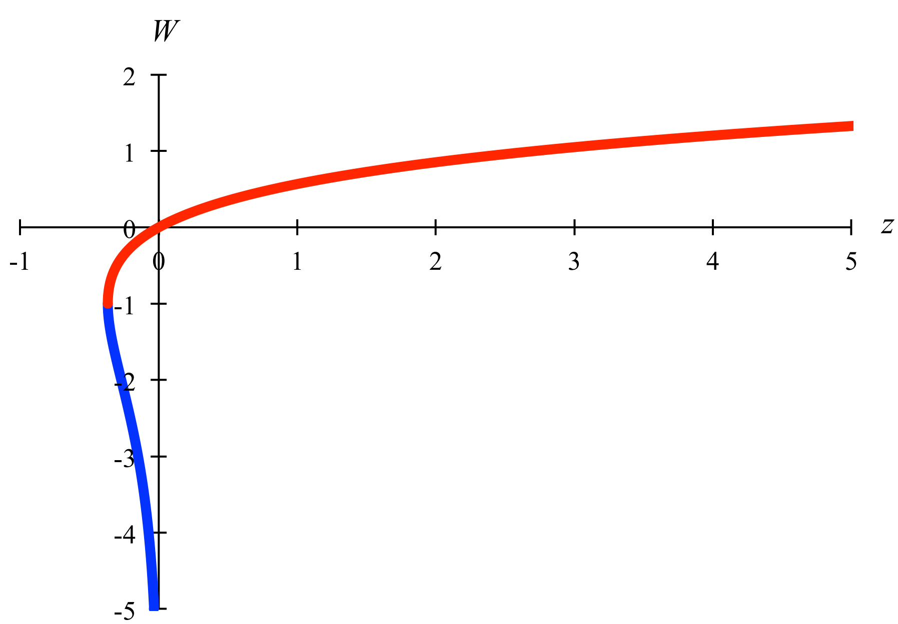
then the equation becomes

$$We^{W} = z \tag{4}$$

whose inverse $W(z)$ defines the Lambert W function. Thus the solution to Eq. (1) is

$$x = -\frac{c}{b} - \frac{1}{a}W\left(-\frac{a}{b}e^{-ac/b}\right). \tag{5}$$

So to finish the problem, we need that function. One way to get it is graphically. Simply compute $z(W)$ for a range from large negative to large positive values and plot it in Excel as W versus z. That results in the following graph.



One sees that the function has two branches: the upper red one with domain $z \geq -1/e \approx -0.36788$ and range $W \geq -1$, and the lower blue one with domain $-1/e \leq z < 0$ and range $W \leq -1$.

For example, suppose one wishes to solve

$$e^x = 2x + 3 \tag{6}$$

so that $a = 1$, $b = 2$, and $c = 3$. Then there are two solutions, given by

$$x = -1.5 - W\left(-0.5e^{-1.5}\right) \approx -1.5 - W(-0.111565). \tag{7}$$

By trial and error in Excel one finds that

$$W(-0.111565) \approx \begin{cases} -3.42394 \text{ for blue branch} \\ -0.236625 \text{ for red branch} \end{cases} \tag{8}$$

so that

$$x \approx 1.92394 \text{ or } -1.373375. \tag{9}$$

Another way to get the solution is using software such as WolframAlpha where the solutions are

$$x = -1.5 - \text{ProductLog}\left[-0.5e^{-1.5}\right] \tag{10}$$

for the principal red branch, and

$$x = -1.5 - \text{ProductLog}\left[-1, -0.5e^{-1.5}\right] \tag{11}$$

for the fully negative blue branch.