

Brightness Temperature of a Laser—C.E. Mungan, Spring 2010

Consider a blackbody which uniformly emits radiation from a plane area A . We suppose that the emission is spectrally filtered using a narrow-bandpass filter of center frequency f and bandwidth Δf . We further suppose the emission is angularly filtered into a cylindrical cone of half-angle θ_{\max} . A power P is measured for this filtered beam. The problem is to find the temperature T of the blackbody from these parameters.

We start from the Planck formula for the spectral energy density of a blackbody,

$$u_f = \frac{8\pi hc^{-3} f^3}{\exp(hf / kT) - 1}, \quad (1)$$

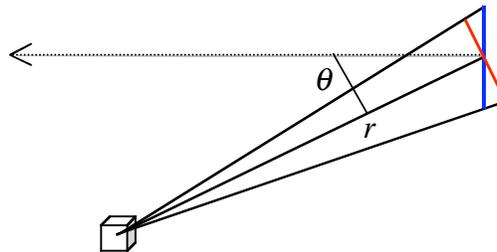
which we integrate across the transmitted spectrum to get the energy density,

$$u = \int_{\Delta f} \frac{8\pi hc^{-3} f^3}{\exp(hf / kT) - 1} df. \quad (2)$$

To get the emitted power P let's imagine that the blackbody is a cavity with a small hole in it of area A . (For example, this cavity could be a gas-filled laser tube, to make contact with the examples discussed at the end of this document.) Then what we need to do is integrate Eq. (2) over two factors: a volume element dV at some point inside the cavity (which we write in spherical coordinates) located a distance $r = ct$ away from the hole such that a photon would just reach the hole in time t , and the fraction F of photons starting at the volume element that are traveling in a direction that would enable them to escape through the hole. The volume element is

$$dV = r^2 \sin\theta dr d\theta d\phi = r^2 \sin\theta c dt d\theta d\phi. \quad (3)$$

To get the escape fraction F , consider the following diagram.



The vertical blue line is an edge view of area A . The dotted line extending leftward is a perpendicular to it, which we will call the z -axis. The volume element is the small cube, located at polar angle θ and radial distance r relative to the hole. Three photons are indicated, striking the top, center, and bottom of the blue area. The red line is the projection of the blue area on to the surface of a sphere of radius r centered on the volume element. We say that the red surface has *projected area* $A_{\perp} = A \cos\theta$ relative to the actual emitting area A in blue. (The factor of $\cos\theta$

here is an expression of *Lambert's cosine law*.) Photons are emitted uniformly across the surface of a sphere of area $A_{\text{sphere}} = 4\pi r^2$ of which the red area is a portion. Therefore

$$F = \frac{A_{\perp}}{A_{\text{sphere}}} = \frac{A \cos \theta}{4\pi r^2}. \quad (4)$$

Multiplying Eqs. (2), (3), and (4) together, integrating over the two angles, and dividing by dt gives the emitted power,

$$\begin{aligned} P &= \frac{A}{4\pi r^2} \int_0^{2\pi} \int_0^{\theta_{\max}} \cos \theta r^2 \sin \theta c d\theta d\phi \int_{\Delta f} \frac{8\pi h c^{-3} f^3}{\exp(hf/kT) - 1} df \\ &= \frac{Ac \sin^2 \theta_{\max}}{4} \int_{\Delta f} \frac{8\pi h c^{-3} f^3}{\exp(hf/kT) - 1} df. \end{aligned} \quad (5)$$

As a check on this expression, note that it reduces to the standard result $P_{\text{BB}} = A\sigma T^4$ if the emission is *not* angularly or spectrally filtered, so that $\theta_{\max} = \pi/2$ and the limits on the frequency integral are from 0 to ∞ . Anyways, the last step is to approximate the integral for a narrow passband, the sine for small cone half-angle, and the exponential for high brightness temperature T (relative to hf/k) so that we can expand it to first order,

$$\boxed{P = 2\pi A c^{-2} f^2 \theta_{\max}^2 kT \Delta f}. \quad (6a)$$

We can alternatively express the result in terms of the center wavelength $\lambda = c/f$ and bandwidth $\Delta\lambda \approx |d\lambda/df| \Delta f = c\Delta f/f^2$ as

$$\boxed{P = 2\pi A c \lambda^{-4} \theta_{\max}^2 kT \Delta\lambda}. \quad (6b)$$

Now let's apply these results to a red HeNe laser ($\lambda = 632.8$ nm) with an output aperture of $A = 1$ mm². First, suppose that $P = 1$ mW, $\Delta f = 1$ GHz, and that the solid angle of the emitted beam is

$$\Omega = \int_0^{2\pi} \int_0^{\theta_{\max}} \sin \theta d\theta d\phi = 2\pi(1 - \cos \theta_{\max}) = 1 \mu\text{sr} \Rightarrow \theta_{\max} = 0.56 \text{ mrad}. \quad (7)$$

In that case, Eq. (6) implies that $T = 1.5 \times 10^{10}$ K, in agreement with *Phys. Teach.* **33**, 497 (1995) after correcting a presumed typo in the exponent for the temperature. Second, suppose that $P = 0.5$ mW, $\Delta\lambda = 1$ pm, and $\theta_{\max} = 0.5$ mrad. Then Eq. (6) implies $T = 1.2 \times 10^{10}$ K as the solution to Baerlein problem 6.15.