The Lily-Pad-Hopping Frog—C.E. Mungan, Fall 2019

There are *n* lily pads in a line in front of a frog standing on an edge of a pond where $n \ge 1$. The frog must eventually jump onto the last lily pad in order to end up close enough to reach the far side of the pond with one final leap. The frog always moves forward by 1 or more lily pads per jump, with equal probability that it will jump forward to any pad that lies ahead of it. How many lily pads does the frog jump onto on average?

Here is a diagram to help understand the situation for the example case of 6 pads:

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Without loss of generality, you can assume the lily pads are equally spaced along the first half of the distance across. The frog is strong enough to jump any distance up to but no more than halfway across the pond. Hence it must get to the middle pad in order to get across.

Let the average number of pads jumped onto be $P(n)$. If $n = 1$ then there is only one pad, at the midpoint of the pond, and the frog must jump onto it so that $P(1) = 1$. Now suppose there is more than 1 pad. On his first jump, the frog has two choices:

• One option is to jump onto the first pad with probability $1/n$. In that case, there are $n-1$ pads ahead of him, so that he will land on $P(n-1)$ more pads on average to finish the trip, for a total of $1 + P(n-1)$ pads visited.

• The other option, with remaining probability $(n-1)/n$, is to bypass the first pad. In that case, the first pad may as well not even be in the pond. He will land on $P(n-1)$ of the remaining pads on average.

To summarize, the average number of pads visited will be

$$P(n) = \tfrac{1}{n}\left[1 + P(n-1)\right] + \tfrac{n-1}{n}\left[P(n-1)\right] = \tfrac{1}{n} + P(n-1)\,. \tag{1}$$

The solution of this recursion relation can be correctly guessed to be

$$P(n) = \sum_{m=1}^{n} \frac{1}{m} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \tag{2}$$

because it has the right starting value for $n = 1$ and the final term added in this series is $1/n$, as Eq. (1) requires. However, this is the divergent harmonic series and so $P(n) \to \infty$ as $n \to \infty$. For large but finite *n*, we can approximate the sum by the integral

$$P(n) \approx \int_1^n \frac{1}{x}\,dx = \ln(n) \tag{3}$$

or a slightly better approximation includes the Euler-Mascheroni constant $\gamma \approx 0.5772$ so that

$$P(n) \approx \ln(n) + \gamma\,. \tag{4}$$