# Local Vertical on Earth—C.E. Mungan, Summer 2000

In Sec. 4.10 of his book "Teaching Introductory Physics," Arnold Arons suggests that a good extra-credit problem is to determine the direction of a freely hanging plumb bob on a spherical, rotating planet. He implies the problem is simple once a free-body diagram is correctly drawn. It neither appears so to me nor is his answer quite entirely correct.
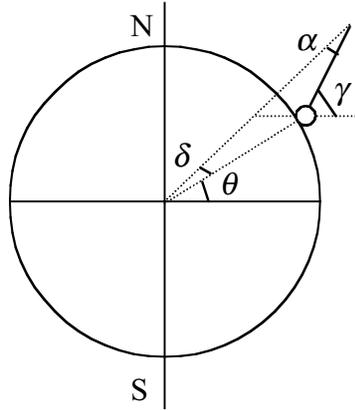

Consider a plumb bob at latitude θ, as sketched above. The problem consists in finding α, the deviation of the local vertical from the radial direction. To do so, it proves necessary to determine the angles γ and δ. The former is easy to find by considering the upper part of the triangle spanned by the string of the plumb bob and the center of the earth, as reproduced below.
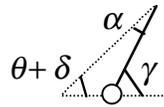
It can be clearly seen from this diagram that

$$\gamma = \theta + \delta + \alpha. \tag{1}$$

To find δ alone, consider the complete triangle instead, as follows, where L is the length of the string and $R = 6380$ km is the radius of the earth.
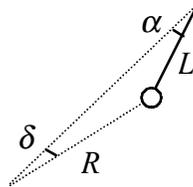

From the law of sines, we have

$$\frac{L}{\sin\delta} = \frac{R}{\sin\alpha} \quad \Rightarrow \quad \delta \cong \frac{L}{R}\alpha \tag{2}$$

since both α and δ are small angles. Furthermore, because the string is much shorter than the radius of the earth, this expression can be substituted into Eq. (1) to obtain

$$\gamma \cong \theta + \alpha \tag{3}$$

to first order in small terms. Now consider the free-body diagram sketched below, where T is the

tension in the string and mg is the weight of the pendulum bob of mass m, with $g = 9.80 \text{ m}/\text{s}^2$ being the gravitational field strength of the planet if it did not rotate.
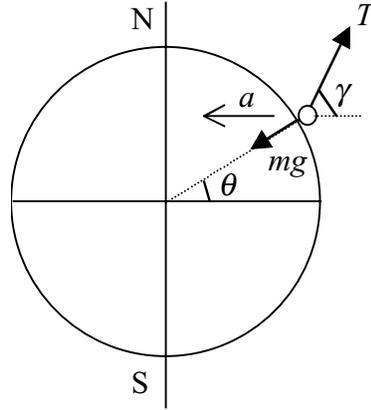
The force components in the north-south direction balance, so that

$$T\sin\gamma = mg\sin\theta \quad \Rightarrow \quad T = mg\frac{\sin\theta}{\sin\gamma}. \tag{4}$$

On the other hand, the components pointing toward the axis of rotation must provide the net centripetal acceleration a, which implies that

$$mg\cos\theta - T\cos\gamma = m\omega^2 R\cos\theta \tag{5}$$

where the angular velocity of the earth is $\omega = 2\pi/T$ with $T = 24 \times 3600$ s. Substituting Eq. (4) into (5) gives

$$1 - \frac{\tan\theta}{\tan\gamma} = \frac{\omega^2 R}{g}. \tag{6}$$

Next, expanding to first order in α, we have

$$\cot\gamma = \frac{\cos(\theta+\alpha)}{\sin(\theta+\alpha)} = \frac{\cos\theta\cos\alpha - \sin\theta\sin\alpha}{\sin\theta\cos\alpha + \cos\theta\sin\alpha} = \frac{1 - \tan\theta\tan\alpha}{\tan\theta + \tan\alpha} \cong \frac{(1-\alpha\tan\theta)(1-\alpha/\tan\theta)}{\tan\theta}. \tag{7}$$

Substituting this into the left-hand side of Eq. (6) gives

$$\alpha(\tan\theta + \cot\theta) \cong \frac{\omega^2 R}{g}. \tag{8}$$

This can be nicely simplified using

$$\frac{1}{\tan\theta + \cot\theta} = \frac{1}{\dfrac{\sin^2\theta}{\cos\theta\sin\theta} + \dfrac{\cos^2\theta}{\sin\theta\cos\theta}} = \cos\theta\sin\theta = \tfrac{1}{2}\sin 2\theta \tag{9}$$

to give the final result

$$\alpha \cong \frac{\omega^2 R}{2g} \sin 2\theta. \tag{10}$$

As expected, the angle of deviation is zero at the poles or equator. The maximum deviation is found at 45° latitude and amounts to 5.9 minutes of arc.