Puzzle of the Lockers—C.E. Mungan, Summer 2021

A large number of lockers l, 2, 3, …, N are all initially closed but N students are going to successively walk by them. Student 1 will toggle lockers l, 2, 3, …; student 2 will then toggle lockers 2, 4, 6, …; student 3 will next toggle lockers 3, 6, 9, …; and so on for all N students. Here, "toggle" means change the state of the locker: open it if it is closed, but close it if is open. After all the students have walked by, which lockers will end up being open?

The answer is all lockers that have an odd number of distinct factors. For example, 7 has only the factors 1 and 7, so it will end up closed: student 1 opens locker 7, but student 7 closes it back up.

Now consider locker number n. If A is a factor, then there must be some B such that $AB = n$. Thus, provided B is distinct from A, every factor has another factor paired with it. Therefore, there will be an even number of factors and locker n will end up closed.

For example, 12 has the paired factors 1 & 12, 2 & 6, and 3 & 4. It will end up getting opened and closed three pairs of times. Thus in the end it will get closed by student 12.

The only way to get an odd number of factors is if B is not distinct from A for one and only one pair of factors of some locker n. In that case $n = A^2$ which means that locker is a perfect square.

For example, 16 has the paired factors 1 & 16 and 2 & 8, and the unpaired factor 4. Thus it will be toggled 5 times and it will end up open.

This puzzle works best if the number N of lockers is equal to the number of students who toggle them. If there are say 3000 lockers but only 2999 students, then the last locker will not get closed, even though 3000 is not a perfect square. On the other hand, if there are 3001 or more students, those extra students will not toggle any lockers and may as well not exist.

Thus the open lockers will be numbers 1, 4, 9, 16, ….