Magic Newton's Cradle—C.E. Mungan, Spring 2015

Consider Newton's cradle consisting of three balls of masses M, αM, and M (in that order) with slight gaps between them so that only two balls collide at a time. If the first ball is pulled aside and released, for what values of α (called "magic mass ratios") will the final result be that the first and second balls come to rest, and the third swings upward to the same height as that from which the first ball was released (as energy conservation requires, since the other balls are at rest)?

Given that both linear momentum and kinetic energy are conserved in every collision, the standard cradle has $\alpha = 1$. In that case, the first ball transfers all its momentum to the second ball in 1 collision, and the second ball in turn transfers all its momentum to the third ball in 1 collision. Since every ball is identical, it only takes 1 collision between ball 2 and each of balls 1 and 3 to accomplish this result.

However, we can get the same final result for certain special values of α less than unity if ball 2 makes multiple collisions with each of the other balls. The above-referenced AJP article works out those magic values for any positive integer by developing a recurrence relation. Here I will verify their solution for the case that ball 2 makes exactly 2 collisions with each of balls 1 and 3.

We need to consider four 1D elastic collisions. Each collision is described by the standard textbook equation for the final velocities determined by simultaneously conserving linear momentum and kinetic energy,

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \tag{1}$$

and likewise for v_{2f} by interchanging subscripts 1 and 2 in this equation.

First collision: We have ball 1 of mass M which can be chosen to equal 1 by appropriate choice of units, and whose initial velocity as it collides with ball 2 can also be chosen to be 1 by appropriate choice of units. Ball 2 then has mass α and zero initial velocity. Equation (1) implies

$$v_{1f} = \frac{1-\alpha}{1+\alpha} \quad \text{and} \quad v_{2f} = \frac{2}{1+\alpha}. \tag{2}$$

Second collision: The middle ball of mass α is now relabeled ball 1 and the third ball of mass 1 is called ball 2. Since their initial velocities are

$$v_{1i} = \frac{2}{1+\alpha} \quad \text{and} \quad v_{2i} = 0, \tag{3}$$

their final velocities become

$$v_{1f} = -\frac{2(1-\alpha)}{(1+\alpha)^2} \quad \text{and} \quad v_{2f} = \frac{4\alpha}{(1+\alpha)^2} \tag{4}$$

where the minus sign indicates the middle ball rebounds backward off the third ball.

Third collision: We revert to the original labels on the balls, so ball 1 has mass 1 and ball 2 has mass α. Hence their initial velocities are

$$v_{1i} = \frac{1-\alpha}{1+\alpha} \quad \text{and} \quad v_{2i} = -\frac{2(1-\alpha)}{(1+\alpha)^2}, \tag{5}$$

so that their final velocities now become

$$v_{1f} = \frac{(1-\alpha)(1-4\alpha-\alpha^2)}{(1+\alpha)^3} \quad \text{and} \quad v_{2f} = \frac{4(1-\alpha)}{(1+\alpha)^3}. \tag{6}$$

We see that the middle ball returns to forward motion after this collision. More importantly, by clever choice of the value of α we can make the first ball come to rest. Since α cannot be 1 (as otherwise ball 1 would have come to rest after the first collision), we require that

$$\alpha^2 + 4\alpha - 1 = 0 \tag{7}$$

which has the unique positive root

$$\alpha = \sqrt{5} - 2 = \sec 36° - 1 \approx 0.236 \tag{8}$$

so that the middle ball must be slightly less than one-quarter as massive as the two end balls.

Fourth collision: The middle ball of mass α is again relabeled ball 1 and the third ball of mass 1 is called ball 2. The initial velocities are then

$$v_{1i} = \frac{4(1-\alpha)}{(1+\alpha)^3} \quad \text{and} \quad v_{2i} = \frac{4\alpha}{(1+\alpha)^2}, \tag{9}$$

and one can check that $v_{1i} > v_{2i}$ using the value of α given by Eq. (6). Equation (1) implies that the final velocities are

$$v_{1f} = \frac{4(\alpha^2 + 4\alpha - 1)}{(1+\alpha)^4} \quad \text{and} \quad v_{2f} = \frac{4\alpha(3 - 2\alpha - \alpha^2)}{(1 + 2\alpha + \alpha^2)^2}. \tag{10}$$

Equation (7) immediately implies that $v_{1f} = 0$. But we can also rewrite Eq. (7) as

$$\alpha^2 = 1 - 4\alpha \tag{11}$$

which can be substituted into the numerator and denominator of Eq. (10) to get

$$v_{2f} = \frac{8\alpha + 8\alpha^2}{4 - 8\alpha + 4\alpha^2}. \tag{12}$$

Substituting Eq. (11) into the numerator and denominator again, one finally obtains $v_{2f} = 1$, thus completing the solution.