Matrix Coefficients for a System of Linear Equations—C.E. Mungan, Spring 2007

Consider the matrix equation $\overleftrightarrow{\mathbf{A}}\vec{\mathbf{x}} = \vec{\mathbf{d}}$ where $\overleftrightarrow{\mathbf{A}}$ is an $n \times n$ matrix, and $\vec{\mathbf{x}}$ and $\vec{\mathbf{d}}$ are $n \times 1$ column vectors. A standard math problem is to find $\vec{\mathbf{x}}$ given $\overleftrightarrow{\mathbf{A}}$ and $\vec{\mathbf{d}}$. The solution is $\vec{\mathbf{x}} = \overleftrightarrow{\mathbf{A}}^{-1}\vec{\mathbf{d}}$, which exists and is unique provided that $\overleftrightarrow{\mathbf{A}}$ is nonsingular, i.e., $\det\overleftrightarrow{\mathbf{A}} \neq 0$.

But now suppose we are interested in finding $\overleftrightarrow{\mathbf{A}}$ given $\vec{\mathbf{x}}$ and $\vec{\mathbf{d}}$. Right-multiplying the matrix equation by $\vec{\mathbf{x}}^t$, a solution is the (singular) matrix $\overleftrightarrow{\mathbf{A}} \equiv \vec{\mathbf{d}}\vec{\mathbf{x}}^t / x^2$ where the superscript "t" denotes the transpose and where the dot product of vector $\vec{\mathbf{x}}$ with itself is $\vec{\mathbf{x}}^t\vec{\mathbf{x}} \equiv x^2$, the square of its magnitude. (We assume that $x \neq 0$, as otherwise the only possibility is the trivial case $\vec{\mathbf{x}} = \vec{\mathbf{d}} = \vec{0}$, the null vector, so that $\overleftrightarrow{\mathbf{A}}$ is completely arbitrary.) In particular, for the eigenvalue problem $\vec{\mathbf{d}} = \lambda\vec{\mathbf{x}}$, this solution becomes the symmetric matrix $\overleftrightarrow{\mathbf{A}} = \lambda\vec{\mathbf{x}}\vec{\mathbf{x}}^t / \vec{\mathbf{x}}^t\vec{\mathbf{x}}$ whose trace is λ and whose determinant is zero.

However, it is easy to see that this solution is not unique. The first row of the matrix equation $\overleftrightarrow{\mathbf{A}}\vec{\mathbf{x}} = \vec{\mathbf{d}}$ is

$$\sum_{i=1}^{n} A_{1i} x_i = d_1 . \tag{1}$$

By assumption, not all of the x_i can be zero; if say $x_j \neq 0$, then we can freely choose A_{1i} for all $i \neq j$, and only the value of A_{1j} is constrained; namely it must be

$$A_{1j} = x_j^{-1}\left[d_1 - \sum_{i=1}^{n}{}' A_{1i} x_i \right] \tag{2}$$

where the prime on the summation indicates that we exclude the case of $i = j$. This fact means that $n-1$ of the coefficients in the first row of matrix $\overleftrightarrow{\mathbf{A}}$ are arbitrary. Since this is true for every row, $n(n-1)$ of the coefficients in the matrix overall are arbitrary; only n of their values are constrained by the matrix equation.

Noting that the following solution for the matrix has $n(n-1)$ independent coefficients, it is therefore the general solution,

$$\overleftrightarrow{\mathbf{A}} \equiv \frac{\vec{\mathbf{d}}\vec{\mathbf{x}}^t}{x^2} + \overleftrightarrow{\mathbf{B}} \quad \text{with} \quad \overleftrightarrow{\mathbf{B}} \equiv \sum_{k=1}^{n-1} \vec{\mathbf{c}}_k \vec{\mathbf{x}}_{\perp k}^t , \tag{3}$$

where $\{\vec{\mathbf{c}}_k\}$ is a set of *arbitrary* $n \times 1$ column vectors and $\{\vec{\mathbf{x}}_{\perp k}\}$ is any *fixed* set of linearly independent vectors spanning the space perpendicular to vector $\vec{\mathbf{x}}$ (so that $\vec{\mathbf{x}}_{\perp k}^t\vec{\mathbf{x}} = 0$ for all k from 1 to $n-1$).

Another way to understand Eq. (3) is that $\overleftrightarrow{\mathbf{B}}$ is the general solution to the equation $\overleftrightarrow{\mathbf{B}}\vec{\mathbf{x}} = \vec{0}$ (for the given vector $\vec{\mathbf{x}}$). Consider the following diagonal matrix of zeroes and ones,

$$\overleftrightarrow{\mathbf{D}} = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \tag{4}$$

and the matrix of column vectors

$$\vec{\mathbf{U}} = \begin{bmatrix} \vec{\mathbf{x}} & \vec{\mathbf{y}}_1 & \cdots & \vec{\mathbf{y}}_{n-1} \end{bmatrix} \tag{5}$$

where $\{\vec{\mathbf{y}}_k\}$ is any *arbitrary* set of vectors in the space perpendicular to vector $\vec{\mathbf{x}}$. (This matrix is unitary if the set is linearly independent.) Now compute $\vec{\mathbf{B}}$ from the similarity transform $\vec{\mathbf{B}} = \vec{\mathbf{U}}\vec{\mathbf{D}}\vec{\mathbf{U}}^t$. Then by construction $\{\vec{\mathbf{x}}, \vec{\mathbf{y}}_k\}$ is a set of eigenvectors (incomplete if $\{\vec{\mathbf{y}}_k\}$ is linearly dependent) that satisfies the equation $\vec{\mathbf{B}}\vec{\mathbf{z}} = \lambda\vec{\mathbf{z}}$ with eigenvalues $\lambda = 0$ when $\vec{\mathbf{z}} = \vec{\mathbf{x}}$ and $\lambda = 1$ when $\vec{\mathbf{z}} = \vec{\mathbf{y}}_k$. It has $n(n-1)$ arbitrary coefficients (specifying the vectors $\vec{\mathbf{y}}_k$) and thus $\vec{\mathbf{B}}$ is the required general solution. This solution for $\vec{\mathbf{B}}$ becomes equal to that defined in Eq. (3) for the special choice $\vec{\mathbf{c}}_k = \vec{\mathbf{y}}_k = \vec{\mathbf{x}}_{\perp k}$.

To illustrate, let's write out a couple of examples of Eq. (3) for the simple case of $n = 2$. If we choose $\vec{\mathbf{c}} = \vec{0}$, then the matrix solution becomes

$$\vec{\mathbf{A}} = \begin{bmatrix} \dfrac{d_1 x_1}{x_1^2 + x_2^2} & \dfrac{d_1 x_2}{x_1^2 + x_2^2} \\ \dfrac{d_2 x_1}{x_1^2 + x_2^2} & \dfrac{d_2 x_2}{x_1^2 + x_2^2} \end{bmatrix} \quad \text{(one solution)}. \tag{6}$$

Another solution, valid if say x_1 specifically is nonzero, is

$$\vec{\mathbf{A}} = \begin{bmatrix} \dfrac{d_1}{x_1} & 0 \\ \dfrac{d_2}{x_1} & 0 \end{bmatrix} \quad \text{(another solution)}, \tag{7a}$$

obtained by choosing

$$\vec{\mathbf{c}} = \frac{x_2}{x_1}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{x}}_\perp = \frac{1}{x_1^2 + x_2^2}\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}. \tag{7b}$$

Note for both solutions (6) and (7) that

$$A_{11} = \frac{d_1 - A_{12} x_2}{x_1} \quad \text{and} \quad A_{21} = \frac{d_2 - A_{22} x_2}{x_1}, \tag{8}$$

in accordance with Eq. (2).