Phase and Group Velocity of Matter Waves—C.E. Mungan, Spring 2017

Consider a beam of particles traveling in free space in the same direction with nonrelativistic speed υ. Find their quantum-mechanical phase velocity υ_p and group velocity υ_g.

The phase speed of a wave is $\upsilon_\text{p} = \omega / k$ where $\omega = 2\pi f$ is the angular frequency and $k = 2\pi / \lambda$ is the angular wavenumber (with *f* and λ the frequency and wavelength). Assume the energy *E* and linear momentum *p* of the particles are given by the Einstein and de Broglie relations as $E = \hbar\omega = hf$ and $p = \hbar k = h / \lambda$ in terms of Planck's constant. (If you wish, you could take these as *definitions* of the frequency and wavelength of the matter wave associated with the beam of particles.) In the nonrelativistic limit such that

$$\gamma \equiv \frac{1}{\sqrt{1-\upsilon^2 / c^2}} \approx 1 \tag{1}$$

to lowest order, then $p = \gamma m\upsilon \approx m\upsilon$ as expected classically where *m* is the (rest) mass of the particles. So far we have

$$\upsilon_\text{p} \approx \frac{E}{m\upsilon} \tag{2}$$

but there is now a choice of two ways to proceed.

Classically *E* is the sum of the kinetic and potential energies. In free space there is no potential energy and thus

$$\upsilon_\text{p} \approx \frac{K}{m\upsilon} = \frac{\frac{1}{2}m\upsilon^2}{m\upsilon} = \tfrac{1}{2}\upsilon \tag{3}$$

which is a strange result, in that there is nothing physically traveling at half the speed of the particles. Nevertheless Griffiths[1] and Mundarain[2] argue that Eq. (3) is correct because it accords with the Schrödinger equation $i\hbar\partial\psi / \partial t = -(\hbar^2 / 2m)\partial^2\psi / \partial x^2$ for a plane wave $\psi(x,t) = A\exp(ikx - i\omega t)$.

Another approach is to start from special relativity to get $E = \gamma mc^2 \approx mc^2$ in which case

$$\upsilon_\text{p} \approx \frac{mc^2}{m\upsilon} = \frac{c}{\upsilon}c \tag{4}$$

which is also strange in that it is superluminal. This answer is valid even for relativistic particles because

$$\upsilon_\text{p} = \frac{E}{p} = \frac{\gamma mc^2}{\gamma m\upsilon} \tag{5}$$

without making any approximations.

We can understand the difference between Eqs. (3) and (4) because the first two terms in the expansion for the relativistic energy are $E = mc^2 + \frac{1}{2}m\upsilon^2$ and putting that into Eq. (2) would give the *sum* of these two answers for the phase speed. In other words, the issue boils down to whether the lowest order approximation to the particle energy is its rest energy or if instead the rest energy should be treated classically as an unimportant zero point and discarded. Presumably the answer may depend on the intended application. For example, Zhou[3] argues that Eq. (5) is necessary to describe neutrino oscillations using the Klein-Gordon equation.

Information is carried at the group velocity. Reasoning as before, we have $\upsilon_{\mathrm{g}} = d\omega / dk = dE / dp$. Classically this expression becomes

$$\upsilon_{\mathrm{g}} \approx \frac{d}{dp}\left(\frac{p^2}{2m}\right) = \frac{p}{m} = \upsilon \tag{6}$$

which, in contrast to the phase velocity, is an intuitively sensible answer. In fact, it remains valid even for relativistic particles because

$$\upsilon_{\mathrm{g}} = \frac{d}{dp}\sqrt{m^2c^4 + p^2c^2} = \frac{c^2 p}{\sqrt{m^2c^4 + p^2c^2}} = \frac{c^2 p}{E} = \frac{c^2\gamma m\upsilon}{\gamma mc^2} = \upsilon \tag{7}$$

which makes it even more satisfying. In particular, for a photon $E = cp$ so that

$$\upsilon_{\mathrm{g}} = \frac{d}{dp}(cp) = c\ . \tag{8}$$

For a photon, one also gets $\upsilon_{\mathrm{p}} = E / p = c$. The key is that for a highly relativistic particle, *E* is linear in *p* so that $dE / dp = E / p \Rightarrow \upsilon_{\mathrm{g}} = \upsilon_{\mathrm{p}}$. The extra factor of 2 in Eq. (3) comes about because *E* is quadratic in *p* for a classical particle.

1. D.J. Griffiths, *Introduction to Quantum Mechanics* (Prentice Hall, Upper Saddle River NJ, 1995), Sec. 2.4.
2. D. Mundarain, “About the nonrelativistic limit of the phase velocity of matter waves,” Eur. J. Phys. **38**, 045402 (2017).
3. S. Zhou, “Phase velocity and rest energy of Schrödinger equation seen from neutrino oscillations,” J. Mod. Phys. **7**, 473 (2016).