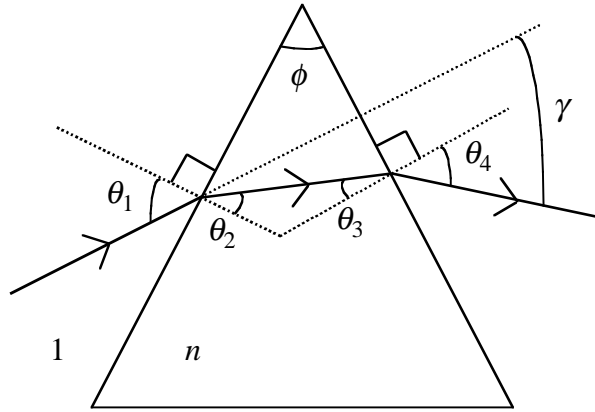# Angle of Minimum Deviation through a Prism—C.E. Mungan, Spring 2001

Prove the well-known fact that one gets minimum deviation for a symmetric beam geometry.

It is easy to see from the geometry of this sketch that the apex angle ϕ is related to the two internal angles of refraction by

$$(90°-\theta_2) + \phi + (90°-\theta_3) = 180° \quad \Rightarrow \quad \phi = \theta_2 + \theta_3, \tag{1}$$

and that the angle of deviation γ is given by

$$\gamma = (\theta_1 - \theta_2) + (\theta_4 - \theta_3) = \theta_1 + \theta_4 - \phi. \tag{2}$$

We get two more equations by applying Snell's law to the first interface,

$$\sin\theta_1 = n\sin\theta_2, \tag{3}$$

and to the second interface,

$$\sin\theta_4 = n\sin\theta_3. \tag{4}$$

But minimum deviation arises when

$$\frac{d\gamma}{d\theta_1} = 0 \quad \Rightarrow \quad \frac{d\theta_4}{d\theta_1} = -1 \tag{5}$$

using Eq. (2). Now differentiate Eq. (4) with respect to θ_1 to get

$$\cos\theta_4 \frac{d\theta_4}{d\theta_1} = n\cos\theta_3 \frac{d\theta_3}{d\theta_1} \quad \Rightarrow \quad -\cos\theta_4 = n\cos\theta_3 \frac{d\theta_3}{d\theta_1} \tag{6}$$

after substituting Eq. (5). Similarly, differentiate Eq. (1) with respect to θ_1 to obtain

$$\frac{d\theta_3}{d\theta_1} = -\frac{d\theta_2}{d\theta_1} \quad \Rightarrow \quad -\cos\theta_2\cos\theta_4 = -n\cos\theta_2\cos\theta_3 \frac{d\theta_2}{d\theta_1}. \tag{7}$$

The second equality follows from substituting the first result into Eq. (6) and pre-multiplying by $\cos\theta_2$. Again, differentiate Eq. (3) with respect to θ_1 and pre-multiply by $\cos\theta_3$ to give

$$\cos\theta_3\cos\theta_1 = n\cos\theta_3\cos\theta_2 \frac{d\theta_2}{d\theta_1}. \tag{8}$$

Addition of Eqs. (7) and (8) results in

$$\cos\theta_3 \cos\theta_1 = \cos\theta_2 \cos\theta_4. \tag{9}$$

Square this equation and substitute Eqs. (3) and (4) to obtain

$$\left(1 - \sin^2\theta_3\right)\left(1 - n^2 \sin^2\theta_2\right) = \left(1 - \sin^2\theta_2\right)\left(1 - n^2 \sin^2\theta_3\right)$$

which simplifies to

$$\left|\sin\theta_2\right| = \left|\sin\theta_3\right|$$

assuming $n \neq 1$. The only solution to this equation which is consistent with Eq. (1) is

$$\boxed{\theta_2 = \theta_3 = \tfrac{1}{2}\phi \quad \text{and so} \quad \theta_1 = \theta_4} \tag{10}$$

using Eqs. (3) and (4). This has the desired symmetry.

Hecht points out a much simpler argument for why θ_1 must be equal to θ_4 at minimum deviation. Namely, if this were not the case, then we could optically reverse the ray to prove that there must be two distinct angles of minimum deviation, which is not true.

It is also worth remarking on one of the many applications of minimum deviation. Note from Eq. (2) that

$$\theta_1 = \frac{\gamma_{\min} + \phi}{2} \tag{11}$$

and from Eq. (3) it follows that

$$\sin\theta_1 = n\sin\frac{\phi}{2} \tag{12}$$

for minimum deviation. Substituting Eq. (11) into (12) gives a means of determining the index of refraction of a prism of known apex angle,

$$\boxed{n = \frac{\sin\left(\dfrac{\gamma_{\min} + \phi}{2}\right)}{\sin\left(\dfrac{\phi}{2}\right)}} \tag{13}$$

from a measurement of the angle of minimum deviation.