Modified Taylor Series—C.E. Mungan, Spring 2008

A 1D Taylor series for the function $f(y)$ expanded about the point y_0 is

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n f}{dy^n} \right|_{y=y_0} (y - y_0)^n \,. \tag{1}$$

Although y_0 is normally considered a constant, this formula is valid for any y_0 and so we can certainly treat it as a variable x. Notice that the derivative is therefore of the function with respect to its argument and is then evaluated at x. For brevity, we can therefore write it as $f^{(n)}(x)$. Equation (1) can therefore be rewritten as

$$f(y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (y - x)^n \,. \tag{2}$$

We now let $y \equiv x + h(x)$ to obtain the desired series,

$$\boxed{f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)} \,. \tag{3}$$

Proper convergence of this series for given functions $f(y)$ and $h(x)$ should be checked using a standard method such as the ratio test.

As an example of the application of this result, suppose that $f(y) = y^{-1}$ and that $h(x) = e^x$. Noting that

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} \,, \tag{4}$$

Eq. (3) becomes

$$\frac{1}{x + e^x} = \sum_{n=0}^{\infty} \frac{(-1)^n e^{nx}}{x^{n+1}} \,. \tag{5}$$

It is easy to verify that this result is correct. Multiplying both sides by x we get

$$\frac{1}{1 + e^x / x} = \sum_{n=0}^{\infty} \left(-\frac{e^x}{x} \right)^n \tag{6}$$

which then becomes immediately recognizable as a geometric series, convergent provided $e^x < |x|$ which is true for approximately $x < -0.57$.

If one substitutes $h(x) \equiv g(x) - x$ into Eq. (3), one obtains

$$\boxed{f\left(g(x)\right)=\sum_{n=0}^{\infty}\frac{(g-x)^n}{n!}f^{(n)}(x)}. \tag{7}$$

For example, if $g(x)=\sin x$ and $f(y)=\sin y$, then since

$$f^{(n)}(x)=\begin{cases}(-1)^{n/2}\sin x & \text{if } n=0,2,4,\ldots\\ (-1)^{(n-1)/2}\cos x & \text{if } n=1,3,5,\ldots\end{cases}, \tag{8}$$

it follows that

$$\begin{aligned}\sin(\sin x)&=\sin x+(\sin x-x)\cos x-\tfrac{1}{2}(\sin x-x)^2\sin x-\tfrac{1}{6}(\sin x-x)^3\cos x+\cdots\\ &=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}[(2n+1)\sin x+(\sin x-x)\cos x](\sin x-x)^{2n}\end{aligned} \tag{9}$$

so that Eq. (7) can be used to derive some unusual identities.