A 1D Taylor series for the function \( f(y) \) expanded about the point \( y_0 \) is

\[
f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f}{dy^n} \bigg|_{y=y_0} (y - y_0)^n.
\]

(1)

Although \( y_0 \) is normally considered a constant, this formula is valid for any \( y_0 \) and so we can certainly treat it as a variable \( x \). Notice that the derivative is therefore of the function with respect to its argument and is then evaluated at \( x \). For brevity, we can therefore write it as \( f^{(n)}(x) \).

Equation (1) can therefore be rewritten as

\[
f(y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (y - x)^n.
\]

(2)

We now let \( y \equiv x + h(x) \) to obtain the desired series,

\[
f(x + h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x) .
\]

(3)

Proper convergence of this series for given functions \( f(y) \) and \( h(x) \) should be checked using a standard method such as the ratio test.

As an example of the application of this result, suppose that \( f(y) = y^{-1} \) and that \( h(x) = e^x \).

Noting that

\[
f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}},
\]

(4)

Eq. (3) becomes

\[
\frac{1}{x + e^x} = \sum_{n=0}^{\infty} \frac{(-1)^n e^{nx}}{x^{n+1}} .
\]

(5)

It is easy to verify that this result is correct. Multiplying both sides by \( x \) we get

\[
\frac{1}{1 + e^x / x} = \sum_{n=0}^{\infty} \left( -\frac{e^x}{x} \right)^n
\]

(6)

which then becomes immediately recognizable as a geometric series, convergent provided \( e^x < |x| \) which is true for approximately \( x < -0.57 \).

If one substitutes \( h(x) \equiv g(x) - x \) into Eq. (3), one obtains
\[ f(g(x)) = \sum_{n=0}^{\infty} \frac{(g-x)^n}{n!} f^{(n)}(x). \]  

(7)

For example, if \( g(x) = \sin x \) and \( f(y) = \sin y \), then since

\[ f^{(n)}(x) = \begin{cases} (-1)^{n/2} \sin x & \text{if } n = 0, 2, 4, \\ (-1)^{(n-1)/2} \cos x & \text{if } n = 1, 3, 5, \ldots \end{cases}, \]

(8)

it follows that

\[
\sin(\sin x) = \sin x + (\sin x - x) \cos x - \frac{1}{2} (\sin x - x)^2 \sin x - \frac{1}{6} (\sin x - x)^3 \cos x + \cdots
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} [(2n+1) \sin x + (\sin x - x) \cos x] (\sin x - x)^{2n}
\]

(9)

so that Eq. (7) can be used to derive some unusual identities.