Moon Tans—C.E. Mungan, Spring 2000

Compare the average luminosities of the sun and full moon at zenith. This was discussed by PHYS–L in December 1999 and in this note I analyze this problem in greater detail.

The bidirectional reflectance distribution function (BRDF) describing surface scattering is defined to be the ratio of the reflected radiance to the incident irradiance. The irradiance is the light power (flux) incident per unit surface area,

$$H_i \equiv \frac{dP_i}{dA}. \tag{1}$$

Be careful to distinguish this from the beam intensity, which is the power per unit perpendicular area of the beam, $I_i \equiv H_i / \cos\theta_i$, where θ_i is the angle between the source direction and a normal to the element of surface dA. On the other hand, the radiance (brightness) is the reflected power per unit solid angle per unit surface area projected into the reflected direction,

$$L_r \equiv \frac{dP_r}{dA \cos\theta_r d\Omega_r}. \tag{2}$$

Thus, the BRDF is

$$f \equiv \frac{L_r}{H_i} = \frac{dP_r}{\cos\theta_r d\Omega_r dP_i}. \tag{3}$$

A Lambertian surface is defined to be one which appears equally bright from any viewing angle, so that

$$H_r \equiv \int L_r \cos\theta_r d\Omega_r = 2\pi L_r \int_0^{\pi/2} \cos\theta_r \sin\theta_r d\theta_r = \pi L_r \tag{4}$$

is the hemispherical exitance. (A common mistake is to assume the answer should be $2\pi L_r$, forgetting the projection factor $\cos\theta_r$ in the above integral.) Consider the situation where the surface is uniformly illuminated from all directions. In that case,

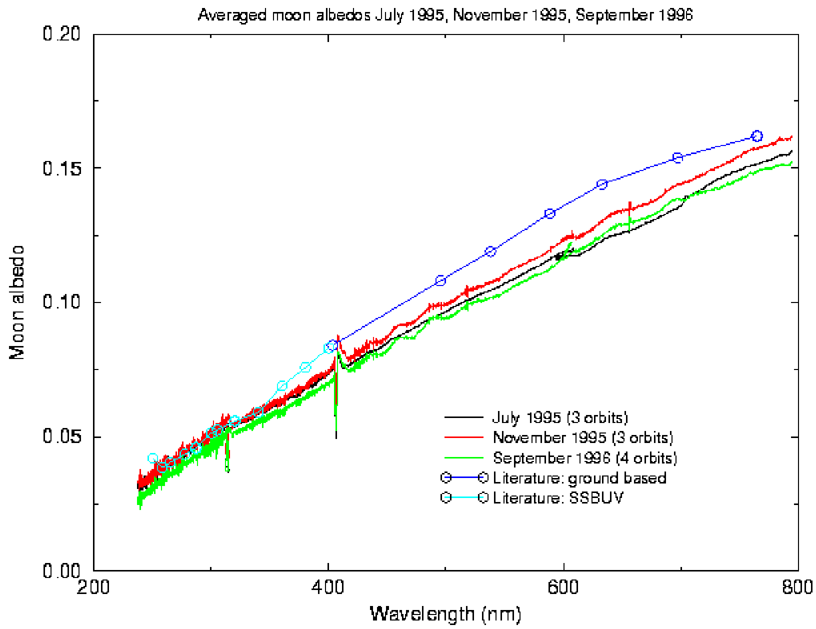
$$\rho \equiv \frac{H_r}{H_i} \quad \Rightarrow \quad f = \frac{\rho}{\pi} \tag{5}$$

where ρ is the albedo or bihemispherical reflectance.

Suppose that the moon were Lambertian. In that case, we see from the above equations that the reflected intensity from some small area on the lunar disk tilted at angle $\theta \equiv \theta_i = \theta_r$ to the direction of the sun (or equivalently of the earth since it is at nadir) is

$$I_r = \frac{\rho}{\pi} I_i \cos\theta \Delta\Omega \tag{6}$$

where $\Delta\Omega \cong A_{\text{det}} / r_{EM}^2$ is the (assumed small) solid angle subtended by some detector of aperture area A_{det} on earth at a distance r_{EM} from the moon. Here, the incident intensity I_i can be taken roughly equal to the earth-based solar constant of 1372 W/m^2; the lunar albedo ρ averaged over the near side varies with wavelength, as can be seen in the graph below taken from recent GOME satellite measurements <http://earth.esa.int/florence/papers/data/dobber/>, but a mid-visible value is about 10%.



According to Eq. (6), the scattered intensity is proportional to the cosine of the angle between the viewing direction and the surface normal. This is known as Lambert's law and answers the following question. Since blackbodies obey Lambert's law in emission, the sun appears uniformly bright across its solar disk, as explained in connection with Eq. (4). (I have a pair of binoculars with exposed x-ray film as filters which you can use to check this claim for yourself.) If the moon were a Lambertian scatterer, would it therefore appear uniformly bright across its disk when full and at the zenith? The answer is no; Eq. (6) tells us that it would be brightest at its center and fall to total blackness at its limbs. The difference between the sun and moon is that the sun emits isotropically while the moon is only illuminated from one fixed direction. (The moon *would* appear uniformly bright if it were illuminated isotropically.)

However, it is a common observation and one validated by quantitative measurements that the moon *does* appear very nearly uniformly bright across its disk. (In fact, there is a coherent backscattering effect, but that's another story I don't have time to discuss here.) Therefore, the moon is *not* Lambertian.

But wait! At the risk of totally confusing you, the fact that it appears uniformly bright means that we may nevertheless model it as a Lambertian disk (rather than a Lambertian hemisphere) by putting $\theta = 0^\circ$ and $A = \pi R_M^2$ where R_M is the lunar radius, so that Eq. (6) becomes

$$P_r = \rho I_i R_M^2 A_{\text{det}} / r_{EM}^2 . \qquad (7)$$

(The reader is invited to check that this result would be multiplied by 2/3 and thus be 33% smaller if we instead took the moon to be a Lambertian hemisphere.)

On the other hand, if we now point our detector directly at the sun at zenith, the emitted solar power entering its aperture would obviously be

$$P_e = I_i A_{\text{det}}. \tag{8}$$

Thus, taking the ratio of Eqs. (7) and (8) we have for the ratio of detected fluxes,

$$\frac{P_{\text{moon}}}{P_{\text{sun}}} = \rho \left(\frac{R_M}{r_{EM}} \right)^2 \tag{9}$$

which equals 1.9×10^{-6} since the ratio of the lunar radius to its distance is well known to be approximately 0.25° converted to radians. This agrees with an estimate based on the known magnitudes of the moon and sun, $M_{\text{moon}} = -12.5$ and $M_{\text{sun}} = -26.8$, so that the ratio of their luminosities is

$$\frac{L_{\text{moon}}}{L_{\text{sun}}} = 10^{0.4(M_{\text{sun}} - M_{\text{moon}})} = 1.9 \times 10^{-6}. \tag{10}$$

Thus to get a moon tan, you simply need to lie outside about a million times longer (allowing another factor of 2 decrease because of the reduced albedo in the UV) than you would have to on Pensacola beach!