Multiply by Four and Reverse—C.E. Mungan, Fall 2017

What four-digit number when multiplied by 4 is equal to the original number with its digits in reverse order?

Designate the number's digits as *ABCD* where *A* is nonzero. When the number is multiplied by 4, the result is even. Reversing all the digits in this result, we conclude that *A* must be even. But *A* cannot be larger than 2, because otherwise the number would no longer remain four digits in length when multiplied by 4. Therefore $A = 2$.

As we multiply the other digits in the number by 4, we carry up to 3 from one digit of the result to the next (because the largest possible multiplication is if we are multiplying a digit of 9 and adding to it a carry of 3 to get 4x9+3 = 39 which results in a carry of 3). Call any particular carry *E* where $E = 0, 1, 2$, or 3.

But we cannot carry more than 1 from $4B$ to $4A$ because otherwise $4A+E$ would no longer remain a single digit. Furthermore this particular carry cannot be 1 because otherwise $4A+E = 9$ which must become *D* when reversed. But in that case $4D = 36$ which when reversed would imply that $A = 6$ which is wrong. We conclude that there is zero carry from $4B$ to $4A$. Thus when the entire number is multiplied by 4, the largest digit of the result is $4A = 8$ which when reversed implies that $D = 8$.

Now given that $4B$ cannot result in a carry, *B* can be no larger than 2. However, $4D = 32$ results in a carry of 3. But $4C+3$ is odd, and therefore so is the second-smallest digit when the entire number is multiplied by 4. Reversing that result implies that *B* must be odd. We conclude that $B = 1$.

Finally when we multiply *C* by 4 and carry 3 from $4D$ the result must end in 1 to equal *B* when reversed. That is, $4C+3$ has the two-digit form $E1$ where *E* is the carry to $4B$. But since $4C+3$ can be no smaller than 3, *E* cannot be zero. Further *E* cannot be 2 because 21, when the preceding carry of 3 is subtracted from it, equals 18 which is not divisible by 4. The two remaining possibilities are that either $E = 1$ when $C = 2$, or $E = 3$ when $C = 7$. But after multiplying the original four-digit number by 4, the second-largest digit becomes $4B+E = 4+E$ which when reversed must equal *C*. Since $4+E$ cannot equal 2 for positive *E*, *C* must be 7.

To summarize, the only number that solves the puzzle is 2178.

By a similar logic, we can find the six-digit number that when multiplied by 4 is equal to the original number with its digits in reverse order. Each of the intermediate carries must be 3, and so the number is 219978. We can add any other pairs of nines that you like in the middle to make reversible numbers with an even number of digits, such as 21999978 and so on.