Multiplying Two Integers—C.E. Mungan, Spring 2020

(see Russian Peasant or Ethiopian Multiplication)

You went to a school that only teaches really new math, whereby it was considered torture to make children memorize the entire multiplication table. So you only learned to multiply integers by two. But being a quick study, you remembered them well enough that you could also do integer division by two by recalling what number multiplied by two would give the starting value. (Integer division means you can only do that for even starting integers. If the starting value is odd, you reduce its value by 1 before dividing, which is equivalent to discarding any remainder when dividing by 2.) How could you use your knowledge to approximate the multiplication of any two positive integers?

Call the integers *N* and *M* labeled such that $N \le M$. The idea is to find how many times *n* you can do integer division of *N* by 2 until you end up with a result of 1. You then simply need to multiply *M* by 2 a total of *n* times. As an example, suppose you want to multiply 19 by 23. We divide 19 by 2 (discarding any remainder) to get 9, then 4, then 2, and then 1 for a total of $n = 4$ divisions. Now we multiply 23 by 2 to get 46, then 92, then 184, and the final estimate of 368. That can be compared to the exact answer of $19 \times 23 = 437$.

The agreement is not great. Can we find the necessary correction terms? Yes, by a method which is easiest to explain by illustrating this example. Make a two-column table and put the given integers in the first row, *N* on the left and *M* on the right. We want to find the product $P = NM$.

<u>Step 1</u>: If *N* is odd, take the number in the right-hand column and add it to our cumulative estimate of *P*. In the present example, $P = 23 + \cdots$ since 19 is odd. Now replace *N* by $N - 1$. If instead *N* is even, then do nothing in step 1.

<u>Step 2</u>: Divide *N* by 2 and multiply *M* by 2. Write the results on the next row down. Call those two new values *N* in the left-hand column and *M* in the right-hand column.

Repeat steps 1 and 2 until you end up with $N = 1$. The final value of *M* next to it terminates the cumulative estimate of *P*. In the present example, $P = \cdots + 368$.

Here is a diagram of the resulting table for the previous example. Replacements are shown in red. Whenever a replacement is made on the left, the value on the right is boxed. Also the last value in the table is boxed. The sum of the boxed values gives *P*.

~~19~~ 18	$\boxed{23}$
~~9~~ 8	$\boxed{46}$
4	92
2	184
1	$\boxed{368}$

In the present example, the final boxed value is our estimate of 368. The other two boxed values are the correction terms, such that the exact product is $P = 23 + 46 + 368 = 437$.

By examining the table, we can see why this method works. If *N* is even, then the product of the entries on the next row down is equal to *NM*. For example, for row 3 we have $N = 4$ and

$4 \times 92 = 2 \times 184$. No information has been lost because we halved *N* and doubled *M*, so their product remained the same. Thus we do not need any correction term when we move from row 3 to row 4. Further, when we get to the last row we always have $N = 1$ on the left and thus our estimate of the product is the value of *M* on the right in the last row of the table. However, if *N* is odd, then we replaced *N* by $N - 1$, and so the original product *NM* of entries in that row became reduced to $(N - 1)M$ which is smaller than the original product by *M*. Thus we need to add *M* back onto our estimate of the product. But that is just the value sitting to the right of our replaced number. For example, in row 2 the original product is $9 \times 46 = 414$ and the reduced product is $8 \times 46 = 368$ so that we lost the boxed value of 46 from the product when we made the replacement of 9 by 8.

Two final comments on this solution:

(i) The reason I specified $N \le M$ was to reduce the number of operations and thus to speed up the solution. However, the algorithm works fine if you put the larger starting integer in the left column instead of in the right column of the first row of the table. In fact, it can give a more accurate solution. For example, the method will give an exact product with no correction terms if the left-hand starting number is an integer power of 2 such as 64.

(ii) Further, if one is interested in accuracy of the approximation without correction terms, it might be advantageous to replace odd values of *N* with a step-up in value to $N + 1$ rather than the step-down value $N - 1$. In that case, the correction term for that row will be negative instead of positive. (That means best accuracy could be attained by sometimes stepping up and other times stepping down, to try to maximize cancellation of the correction terms.) For example, when multiplying 31 and 5, a better approximation to the exact answer of $31 \times 5 = 155$ is to step up once to $32 \times 5 = 160$ (with a single correction of –5) rather than having to step down four times to end up with the equivalent of $16 \times 5 = 80$ (with correction terms of 5, 10, 20, and 40). In general, we want to converge as quickly as possible onto a value of *N* in the left entries of the table that is an integer power of 2, because there will be no correction terms from then on (and the correction terms rapidly grow in value as one progresses down the table).