My Number is 136—C.E. Mungan, Spring 2018

Logic Puzzle: Ana and Boris know that possibly identical positive integers, a and b, have been chosen by a third party. Their product has been written on Ana's forehead and their sum has been written on Boris's forehead. Each can see the other's number but not his or her own. They will announce the number on their own forehead the instant they know it for sure. Some time after each sees the number on the other's forehead, one says to the other, "I can see why you haven't announced your number, because there is no way you can know it for sure based on the observational data." The other responds, "Thanks! I now know my number is 136." *What number is on Ana and what number is on Boris?*

Answer: Ana made the statement to Boris, and Ana has the number 135 on her forehead.

Explanation: Suppose we require (without loss of generality) that $a \geq b$. If a and b were both equal to 1, then both persons would immediately have figured out that Ana has 1 on her forehead and Boris has 2 on his forehead. Therefore Ana must have a number larger than 1 on her forehead. However, if there were a prime number p on Ana's forehead, then Boris would immediately figure out that the number on his forehead is $p+1$. In contrast, if there were a composite number on Ana's forehead, it would never be possible for Boris to figure out for sure what number must be on his forehead based only on his observation of Ana's number. Thus, Ana knows that Boris would only have a *chance* to figure out his number initially if the number on Boris's forehead is the sum of a prime number and 1. (Seeing such a number on Boris's head does not *guarantee* that Boris will figure out his number, because it does not *guarantee* that a prime number is on Ana's head. But if the number on Boris's forehead is *not* the sum of a prime number and 1, then he has *zero chance* of figuring out his number initially.)

To conclude, when Ana makes her statement to Boris, she is telling him that the number $a+b$ on his forehead is *not* the sum of a prime number and 1. However, suppose that the number on Ana's forehead *happens* to be 135. When Boris initially saw that number, he immediately factored it into all possible integer products ab with $a \geq b$, and he then summed them together as $a+b$ and determined whether or not each sum was of the form of a prime number plus 1, obtaining the results tabulated below (with b arranged in increasing value down the table).

a	b	$a+b$	prime + 1?
135	1	136	no
45	3	48	yes
27	5	32	yes
15	9	24	yes

Consequently, Boris is able to deduce from Ana's statement that the first row of this table is applicable and hence announces that the number on his forehead is 136.

There is only one step left. We need to prove that if any number other than 135 had been on Ana's forehead, Boris would *not* have been able to announce that his number is 136. We are given that $a+b=136$ (because we are told that that is what Boris announces) but it cannot be true that $b=1$ (because otherwise the only possibility would be that $a=135$, which reproduces the above solution). In that case, there would have to be a lower row in a table constructed like the above one which has entries: $a \le 134$, $b \ge 2$, $a+b=136$, no. However, the first row in such a table would have entries: ab, 1, $ab+1$, no. (As a concrete example, if Boris saw that Ana's number is 268, then one possibility is that $a=134$ and $b=2$ so that his own number is 136 which is not the sum of a prime number and 1. However, another possibility is that $a=268$ and $b=1$ so that his number is 269 which is also not the sum of a prime number and 1.) Thus there are two least two rows in the table which are *not* the sum of a prime number and 1. In that case, Ana's information is *not* enough additional information for Boris to be able to figure out what number is on his forehead.

To summarize the proof, the preceding shows that if Boris sees that Ana's number is 135 and he knows that his own number is not the sum of a prime and 1, then he can definitively conclude that his own number must be 136. Further, if Boris sees that Ana's number is anything other than 135, then he cannot definitively conclude that his own number is 136, even given that his number cannot be the sum of a prime and 1. Thus, the only possible scenario under which Boris, after seeing Ana's number and learning from her that his own number is not the sum of a prime and 1, can announce for sure that his own number is 136 occurs if and only if Ana's number is 135.

Reference: Gerry White in the online solution blog for the *Wall Street Journal* Math Puzzle of 15 March 2018