A Numerical Puzzle Involving Sums of Integer Powers—C.E. Mungan, Spring 2023

You know the sum of two numbers. You also know the sum of their cubes. What is the sum of their squares? Here are three additional questions one could ask: Is there always a unique solution to this question? If so, what is a formula for the answer? If not, what happens?

Let the two numbers be A and B, which in general are complex. Defining $x \equiv A + B$ and $y \equiv A^3 + B^3$, the goal is to find $z \equiv A^2 + B^2$. We can factor y as

$$y = (A + B)(A^2 - AB + B^2) \tag{1}$$

and thus

$$\frac{y}{x} = z - AB \tag{2}$$

which is well-defined provided that $x \neq 0$. Now notice that

$$x^2 = z + 2AB\,. \tag{3}$$

Solve Eq. (3) for AB and substitute it into Eq. (2) to get the solution

$$\boxed{z = \frac{2y + x^3}{3x}} \tag{4}$$

which is unique and is again well-defined provided that $x \neq 0$. The latter condition implies that $A \neq -B$ which further implies that $y \neq 0$. Note that $A = -B$ specifically leads to $x = 0$, $y = 0$, and $z = 2A^2$ which, if A is in general complex, means that z can take on any value you like.

Thus the answers to the questions in the first paragraph are: There is always a unique solution, given by Eq. (4), provided that the sum of the two numbers is nonzero. On the other hand, if their sum is zero, then the sum of their squares can take on any complex value z you like, by choosing the two numbers to be $\pm\sqrt{z/2}$.

Here are some examples:

1. If $A = B = 1$ then $x = y = z = 2$.
2. If $x = 8$ and $y = 20$ then $z = 23$ which is real despite the fact that A and B are complex and equal to $4 \pm 3i/\sqrt{2}$. In general A and B are equal to

$$\frac{x}{2} \pm \sqrt{\frac{4y - x^3}{12x}} \tag{5}$$

provided that $x \neq 0$.

3. If $x = y = 0$ and we want say $z = -8$ then A and B are equal to $\pm 2i$.
4. From Eq. (4), $z = 0$ if and only if $A = \pm iB$, such as $A = 1$ and $B = i$.