## Optical Length of a Day—C.E. Mungan, Spring 2006

Serway and Jewett point out in Quick Quiz 35.4 that the optical length of a day including the effect of atmospheric refraction is longer than the geometric length one would get in its absence. In this note, I make a crude estimate of the extra amount of daylight this produces.

Let's assume the observer is on the equator and neglect earth's axial tilt relative to the ecliptic. Also assume the earth is flat without obstructions extending to the east and west horizons. The simplest model is to take the atmosphere to be a homogeneous spherical shell of air with a thickness of $h = 30$ km and an index of refraction of $n = 1.000\,293 \equiv 1 + \Delta n$, surrounded by vacuum. Here is a view seen from above earth's north pole, with the observer at the dot at the top of the earth.
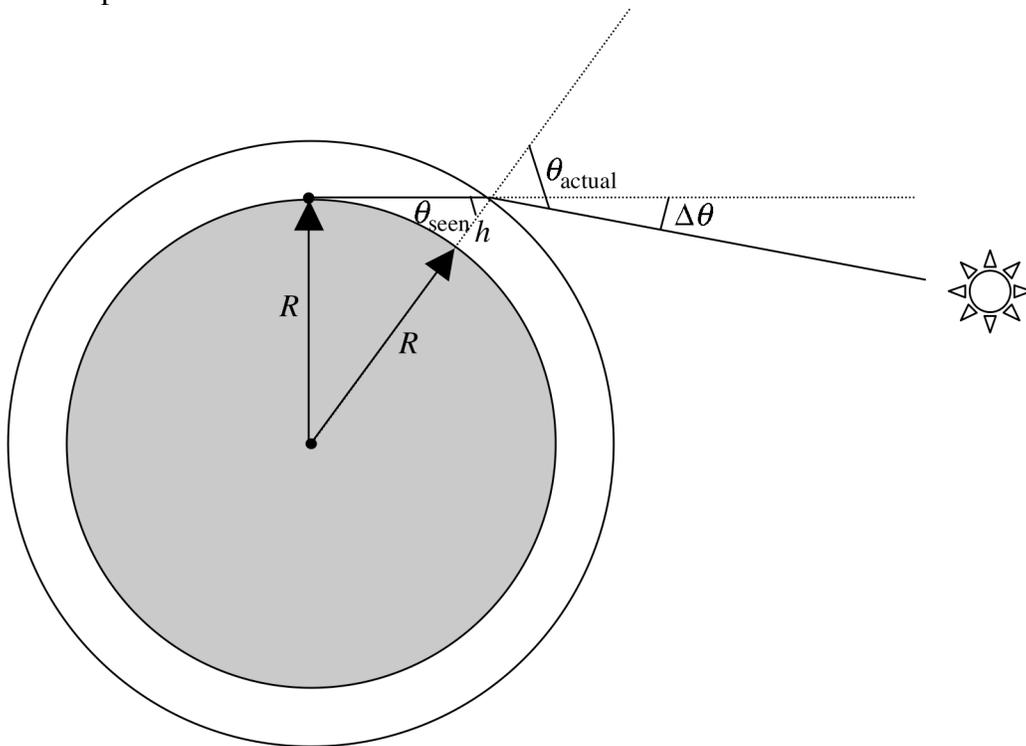
From Snell's law and the geometry of this diagram, we see that

$$\sin\theta_{\text{actual}} = n\sin\theta_{\text{seen}} = n\frac{R}{R+h} \tag{1}$$

which is well-defined because the third expression is less than unity for the values given above together with earth's radius of $R = 6370$ km .

Now notice that

$$\sin\theta_{\text{actual}} = \sin(\theta_{\text{seen}} + \Delta\theta). \tag{2}$$

Substitute the middle expression in Eq. (1) into the left-hand side, and expand the right-hand side using the double-angle formula assuming the deviation angle $\Delta\theta$ is small, to obtain

$$n\sin\theta_{\text{seen}} \cong \Delta\theta\cos\theta_{\text{seen}} + \sin\theta_{\text{seen}} \tag{3}$$

which can be rewritten as

$$\Delta\theta = \Delta n\tan\theta_{\text{seen}}. \tag{4}$$

But again examining the geometry of the triangle in the preceding diagram, we see that

$$\tan\theta_{\text{seen}} = \frac{R}{\sqrt{(R+h)^2 - R^2}} = \frac{1}{\sqrt{\left(1+\dfrac{h}{R}\right)^2 - 1}} \cong \sqrt{\frac{R}{2h}}. \tag{5}$$

Substituting this into Eq. (4) implies that $\Delta\theta = 10'$ of arc, which is about a factor of 3 smaller than a more exact model which includes the altitude variation of the index of refraction.[1] Hence, a day is lengthened by this extra view of the sun at both sunrise and sunset. That is, the optical length of the day is longer than the geometric length by

$$\Delta t = \frac{2\Delta\theta}{\omega} = \frac{\Delta n}{\omega}\sqrt{\frac{2R}{h}} = 83\text{ s}, \tag{6}$$

or a bit over 4 minutes if we triple it,[2] where $\omega = 2\pi/(24\times 3600\text{ s})$ is earth's angular speed.

Endnotes

1. M.E. Thomas and R.I. Joseph, "Astronomical Refraction," APL Technical Digest **17**, 279 (1996), online at <http://techdigest.jhuapl.edu/td1703/thomas.pdf>.

2. Ray Lee calculates $\Delta t = 268$ seconds using the atmospheric model of P.K. Seidelmann (ed.), *Explanatory Supplement to the Astronomical Almanac* (University Science Books, Mill Valley CA 1992), p. 144.