Oscillatory Motion with and without Damping and Driving—C.E. Mungan, Fall 2013

Here I briefly review the harmonic motion of a mass *m* on a spring *k* with or without linear damping of drag coefficient *b* and a driving force of amplitude F_0 and angular frequency *ω*. Three useful combinations of constants are the natural angular frequency

$$\omega_0 \equiv \sqrt{\frac{k}{m}} \tag{1}$$

in rad/s, the damping coefficient

$$\gamma \equiv \frac{b}{2m} \tag{2}$$

in 1/s, and the driving acceleration amplitude

$$f \equiv \frac{F_0}{m} \tag{3}$$

in m/s^2. All three cases below proceed similarly: Write down Newton's second law (N2L), then write down its known solution, and finally verify that solution by substituting it back into N2L.

Case I: Undriven, undamped oscillations

The only force is Hooke's law so that N2L becomes

$$-kx = ma \quad \Rightarrow \quad \frac{d^2x}{dt^2} = -\omega_0^2 x \tag{4}$$

whose solution is

$$x(t) = A\cos(\omega_0 t + \phi) \tag{5}$$

as can be checked in one's head by substituting it back into Eq. (4). Since N2L is a second-order differential equation, there are two constants of integration, the position amplitude *A* and the phase constant *ϕ*. Note that some texts prefer to use a sine in place of the cosine function in Eq. (5) or to put a minus sign instead of the plus sign in front of *ϕ*. Using the form given here, *ϕ* can alternatively be thought of as the initial value of the phase, i.e., the argument of the cosine function at zero time. Equation (5) can be rewritten using the double-angle formula as

$$x(t) = (A\cos\phi)\cos\omega_0 t + (-A\sin\phi)\sin\omega_0 t\ . \tag{6}$$

For example, if we pull the mass so that it stretches the spring and then release the mass from rest, the oscillator will start as a pure cosine function $x = A\cos\omega_0 t$ with $\phi = 0$. On the other hand, if we give the mass an impulsive push in the +*x* direction when it is at its equilibrium position, the oscillator will start as a pure sine function $x = A\sin\omega_0 t$ with $\phi = -\pi/2$.

Case II: Undriven, underdamped oscillations

Now N2L becomes

$$-kx - b\upsilon = ma \quad \Rightarrow \quad \frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0 \tag{7}$$

whose solution is

$$x(t) = Ae^{-\gamma t}\cos(\omega_1 t + \phi) \tag{8}$$

where

$$\omega_1 \equiv \sqrt{\omega_0^2 - \gamma^2}\ . \tag{9}$$

To verify this solution, differentiate Eq. (8) twice and collect together the cosine and sine terms. The first derivative is

$$\frac{dx}{dt} = -\gamma Ae^{-\gamma t}\cos(\omega_1 t + \phi) - \omega_1 Ae^{-\gamma t}\sin(\omega_1 t + \phi) \tag{10}$$

and the second derivative is

$$\frac{d^2x}{dt^2} = (\gamma^2 - \omega_1^2)Ae^{-\gamma t}\cos(\omega_1 t + \phi) + 2\gamma\omega_1 Ae^{-\gamma t}\sin(\omega_1 t + \phi)\ . \tag{11}$$

Substitute Eqs. (8), (10), and (11) into (7) to get

$$\left[\gamma^2 - \omega_1^2 - 2\gamma^2 + \omega_0^2\right]Ae^{-\gamma t}\cos(\omega_1 t + \phi) + \left[2\gamma\omega_1 - 2\gamma\omega_1\right]Ae^{-\gamma t}\sin(\omega_1 t + \phi) = 0\ . \tag{12}$$

The contents of the second square brackets are obviously zero, as are the contents of the first set when one substitutes Eq. (9) into them.

Case III: Sinusoidally driven, damped oscillations

Choosing the zero of time so that the driving force begins as a cosine, N2L becomes

$$F_0\cos\omega t - kx - b\upsilon = ma \quad \Rightarrow \quad \frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = f\cos\omega t \tag{13}$$

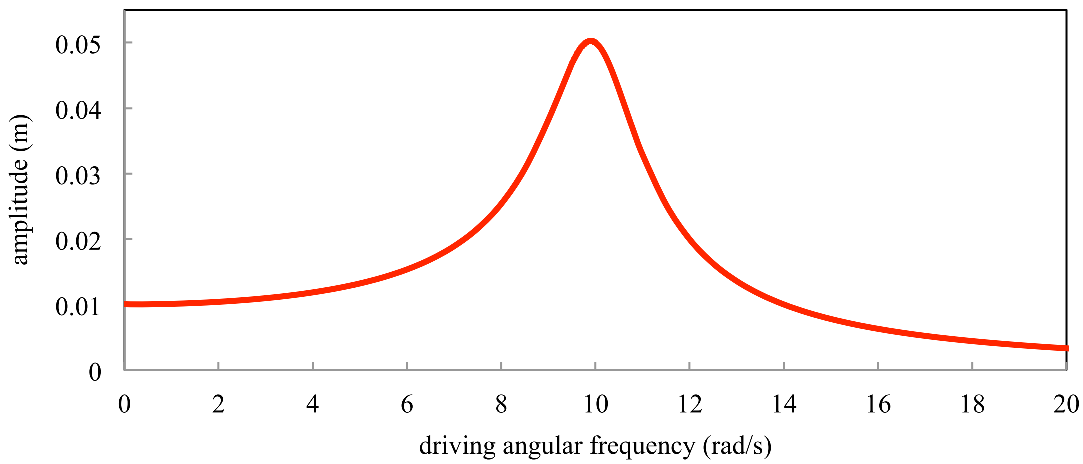
whose steady-state solution is

$$x(t) = A\cos(\omega t - \phi) \tag{14}$$

where

$$A(\omega) = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}} \tag{15}$$

has the usual resonance shape plotted below using Excel for the parameter values $\omega_0 = 10$ rad/s, $\gamma = 1/\text{s}$, and $f = 1\ \text{m/s}^2$. The amplitude is 1 cm at $\omega = 0$, it is 5 cm at $\omega = 10$ rad/s, and it falls to zero as $\omega \to \infty$.



The amplitude peaks when the argument of the square root in the denominator of Eq. (15) is a minimum, namely at an angular frequency ω_2 given by

$$\frac{d}{d\omega}\left[(\omega_0^2-\omega^2)^2+(2\gamma\omega)^2\right] = -4\omega(\omega_0^2-\omega^2)+8\gamma^2\omega = 0 \quad\Rightarrow\quad \omega_2 = \sqrt{\omega_0^2-2\gamma^2} \tag{16}$$

which follows a memorable trend with Eqs. (1) and (9). For the plot values above, $\omega_2 \approx 9.9$ rad/s which is only slightly smaller than ω_0.

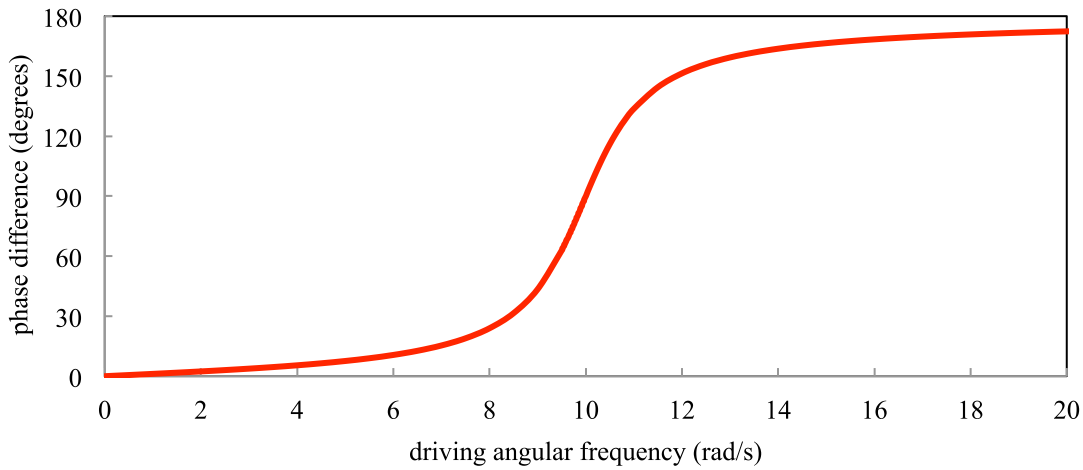
A minus sign was introduced in front of ϕ in Eq. (14) in order to keep that phase constant always positive. It is the phase lag of the mass oscillating as $\cos(\omega t - \phi)$ relative to the driver oscillating as $\cos\omega t$. That is, the mass increasingly cannot keep step with the driver as its frequency of oscillation increases. Specifically, ϕ monotonically rises from 0 (when the mass and driver are in phase) to π (when they are exactly out of phase) as ω increases from 0 (dc) to ∞. It is given by the inverse tangent

$$\phi(\omega) = \tan^{-1}\frac{2\gamma\omega}{\omega_0^2-\omega^2} \tag{17}$$

which passes through $\pi/2$ when $\omega = \omega_0$. Equation (17) implies that

$$\cos\phi = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+(2\gamma\omega)^2}} \quad\text{and}\quad \sin\phi = \frac{2\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+(2\gamma\omega)^2}} \tag{18}$$

as can be verified by noting that these relations satisfy $\sin\phi/\cos\phi = \tan\phi$ and $\cos^2\phi + \sin^2\phi = 1$. Equation (17) is plotted at the top of the next page using the same parameter values as for the previous graph.



To check the solution, again differentiate Eq. (14) twice and collect together cosine and sine terms, except this time use the double-angle formulas to decompose ωt from ϕ in the phases because the driving force does not have the latter constant in its expression. That is, write Eq. (14) as

$$x = A(\cos\omega t\cos\phi + \sin\omega t\sin\phi)\,. \tag{19}$$

Its first derivative is

$$\frac{dx}{dt} = -A\omega(\sin\omega t\cos\phi - \cos\omega t\sin\phi) \tag{20}$$

and its second derivative is

$$\frac{d^2x}{dt^2} = -A\omega^2(\cos\omega t\cos\phi + \sin\omega t\sin\phi)\,. \tag{21}$$

Substitute Eqs. (19), (20), and (21) into (13) to get

$$\left[A(\omega_0^2-\omega^2)\cos\phi + 2A\gamma\omega\sin\phi - f\right]\cos\omega t + \left[(\omega_0^2-\omega^2)\sin\phi - 2\gamma\omega\cos\phi\right]A\sin\omega t = 0\,. \tag{22}$$

The contents of both square brackets become zero when we substitute Eqs. (15) and (18) into them.

The velocity of the oscillating mass is

$$\upsilon(t) = -A\omega\sin(\omega t - \phi) \tag{23}$$

and thus the velocity amplitude is

$$\upsilon_0 = A\omega = \frac{f}{\sqrt{\left(\dfrac{\omega_0^2-\omega^2}{\omega}\right)^2 + 4\gamma^2}} \tag{24}$$

using Eq. (15). The maximum value of this amplitude occurs when the term in parentheses is zero, that is when the driving frequency equals the natural frequency $\omega = \omega_0$, in contrast to the angular frequency ω_2 at which the position amplitude is a maximum. We define "resonance" to occur when the velocity amplitude rather than the position amplitude is a maximum, because that is when the time-averaged power P_{avg} transferred from the driver to the oscillator is maximized. This result follows from the fact that

$$P_{\text{avg}} = (F\upsilon)_{\text{avg}} = \left[-F_0 A\omega \cos\omega t \sin(\omega t - \phi)\right]_{\text{avg}} \tag{25}$$

has a maximum value of $F_0 A\omega / 2$ when $\phi = \pi / 2$. As mentioned after Eq. (17), that phase difference occurs when $\omega = \omega_0$. Although we lose a bit of position amplitude by driving at ω_0 instead of ω_2, we are compensated by the increased velocity amplitude. To understand this behavior intuitively, imagine you are standing near the bottom point of a child on a moving swing. As the swing passes you at $x = 0$ and $\upsilon = \upsilon_0$, you want to push on it with maximum force $F = F_0$ in order to efficiently drive the swing. Thus the force needs to be at a peak when the oscillator passes through its zero position, i.e., the driver and oscillator are 90° out of phase with respect to each other at resonance.