Prove that a parabolic mirror brings collimated incident light to a single focal point. In contrast, determine how much aberration is produced in the focal plane of a spherical mirror.

Let the shape of the parabolic mirror be defined by $y = mx^2$. Then the slope of the tangent line can be expressed either as the derivative or in terms of the angle of incidence $\theta$, so that $\tan \theta = 2mx$. But from the above diagram it is clear that the focal length is

$$f = y + \frac{x}{\tan 2\theta}.$$ 

Thus, by using the trig identity

$$\tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta} = \frac{4mx}{1 - 4m^2x^2},$$

one finds that

$$f = mx^2 + \frac{1 - 4m^2x^2}{4m} = \frac{1}{4m}.$$ 

This is independent of $x$, thus proving the assertion. It also gives a useful relation between the focal length and shape of a parabolic mirror.

Incidentally, the line $y = -f$ is called the directrix. It has the property that it is just as far from any point on the parabola as the focus is. That is

$$\sqrt{(f - y)^2 + x^2} = y + f,$$

as follows by squaring both sides and substituting $y = x^2 / 4f$.

Now, a spherical mirror has a stronger curvature than a parabolic mirror of the same focal length. (After all, it curves all the way around on itself.) Thus the tangent has a steeper slope and our fiducial light ray crosses the focal plane to the left of the principal axis, as sketched below.
The equation for a circle of radius $r$ with the origin at its bottom edge is $x^2 + (y - r)^2 = r^2$, so that

$$y = r - \sqrt{r^2 - x^2}.$$  

To find $f$, we expand this to lowest order in $x$ and match its curvature to that of a parabola,

$$y \rightarrow \frac{x^2}{2r} \equiv \frac{x^2}{4f} \Rightarrow f = \frac{1}{2} r$$

which is a well-known result. Next, we again set the derivative of the shape to the tangent of the angle of incidence,

$$\tan \theta = \frac{dy}{dx} = \frac{x}{\sqrt{r^2 - x^2}}.$$

From the diagram we see that the deviation $d = (f - y)\tan 2\theta - x$. Using the same double-angle trig identity as before and inserting the above three set-off equations leads to the final result

$$d = x \frac{1 - \sqrt{1 - (x/r)^2}}{1 - 2(x/r)^2}.$$  

More practically, suppose that an incident beam of diameter $D \equiv 2x$ centered about and parallel to the principal axis illuminates the mirror. Then the diameter of the blur spot will be

$$2d \equiv \frac{D^3}{8r^2},$$

where I assumed that $D$ is much smaller than the diameter of curvature of the mirror. For example, if $D = f$ (which is probably tighter of a focus than one would ever use and implies that $D$ is 25% of the diameter of curvature) then the blur spot diameter is 3.6% of $D$ according to the exact formula and 3.1% of $D$ according to this approximation.