Parallelogram Numbers Divisible by Eleven—C.E. Mungan, Summer 2019

Rich Downey proposes considering a 3×3 grid of the digits 1 to 9 such as on a telephone keypad or the numeric keypad on extended computer keyboards, as shown below.

7	8	9
4	5	6
1	2	3

If you construct a parallelogram connecting any four entries in this table and go around that parallelogram either clockwise or counter-clockwise starting from any digit, the resulting four-digit number is divisible by 11, such as 1562 or 7931. That can be seen using the "division by 11" rule, whereby if one adds up the digits of a number with alternating signs, the number is divisible by 11 if the resulting sum is equal to zero modulo 11. In our two examples here, $+1-5+6-2=0$ and $+7-9+3-1=0$. It is clear that this will always happen for parallelogram numbers because we move horizontally along one edge of the parallelogram with a positive increment (1 in the first example and 2 in the second) and with a canceling negative increment along the opposite edge. The proof of this rule follows from the fact that successive powers of 10 modulo 11 are equal to alternating powers of –1:

$$1 \pmod{11} = +1, \; 10 \pmod{11} = -1, \; 100 \pmod{11} = +1, \text{ and so on.}$$

That idea implies that one can also create grids that skip by increments other than 1 (say even numbers along one row and odds along another), including negative values (such as reversing the grid above left to right, or even top to bottom for that matter).

Nick Frigo points out that the parallelogram need not have edges parallel to the top and bottom edges of the table: 2486 works for example. Further, the parallelograms can be degenerate, such as 2882. Our rule covers these two examples: the horizontal increment between digits is 1 and 0, respectively.

I note that we can include missing digits (such as 0 above) and even repeat digits by extending the table above sideways, such as below.

4	5	6	7	8	9
1	2	3	4	5	6
9	X	0	1	2	3

The entries are base 11 and so we have an extra digit X before we wrap around back to 0. One way to think of X is as the Roman numeral for "10." For example, 230X means 2310. This means we include some five-digit numbers, such as X340 which means 10340 and three-digit numbers such as 0671 which means 671.

Now if we extend the table far enough, it is clear that we will only need two rows. In that case we arrive at a grid which generates *all* possible numbers up to four digits that are divisible by 11 without repeating any, as follows.

									0	1	2	3	4	5	6	7	8	9									
2	3	4	5	6	7	8	9	X	0	1	2	3	4	5	6	7	8	9	X	0	1	2	3	4	5	6	7

Here the X has a different meaning. It is a square on which you are not allowed to step, like an exploding mine in the old computer game "Minesweeper." The rules of this grid are as follows:

1. Start at any green digit you like.

2. Proceed next to any blue digit you like.
3. Then proceed to another blue digit, possibly the same as the one you chose in step 2.
4. You have now chosen 3 vertices which defines the parallelogram. Proceed to the final digit, which will necessarily be either a green or red digit, but keeping in mind you must start over if you land on either red X.

The reason for this limitation about X is both to avoid creating any five-digit numbers and to avoid repeating four-digit numbers. (For example, 131X means 1320 which can already be generated by a different parallelogram.) Note that the length of the red segments has been chosen so that we can just land on their ends for the two most elongated parallelograms 0902 and 9097.

We can check that every parallelogram on this grid that follows these four rules generates a number that is divisible by 11 with no repetitions by explicitly counting the possibilities. There are 10 choices for the first green digit. Then there are 10 choices for the second blue digit. Then there are 10 choices for the third blue digit. There is then no freedom for the final digit. However, we must reduce the resulting 1000 choices by the number N of parallelograms that end up on an X. How large is N? We have 10 choices for the first green digit of an "exploding" parallelogram. We cannot go up vertically to the blue digit above it. However, there is exactly one exploding parallelogram for every other choice of blue digit (i.e., 9 choices for the second digit). For example, starting at 0 we get exploding parallelograms for 010X, 021X, 032X, and so on. Thus $N = 90$, so that there are 910 numbers with up to four digits that are divisible by 11. Sure enough, that is correct because $9999/11 = 909$ and we have to add on the legal number 0000 according to the preceding four rules.

Another way to summarize things is to say we generate numbers with digits *ABCD* (some or all of which can be zero) according to the following two relations: $B = A + S$ where S is any integer (positive, negative, or zero) such that B remains in the range from 0 to 9; and $D = C - S$ (mod 11). We exclude the result if D ends up being 10, corresponding to the X parallelograms above. For example $A = 9, S = -7, C = 5, D = 12 \pmod{11} = 1$ gives 9251. We again have 10 choices for A, 10 for B, and 10 for C, which we have to reduce by N.

We can deduce some things from this summary or the colored grid above. For example, if the second and third digits are identical, then the parallelogram is necessarily degenerate, in which case the number must be of the form *ABBA*. Here A and B can be any digits from 0 to 9, and both may be identical.