

An Easy Method for Partial Fraction Decomposition—C.E. Mungan, Spring 1997

Jim Marsh came up with the idea behind the following method, which is best illustrated by example.

Suppose we wish to decompose

$$f(z) = \frac{3z}{(z-1)^2(z-2)^2}.$$

We first decompose the function where all of the powers in the denominator are unity and where the numerator is unity,

$$g(z) = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}.$$

Claim: We can immediately obtain A and B from the appropriate residues of $g(z)$.

[Proof: With full generality, we can write $g(z) = \sum_i \frac{a_i}{z-z_i}$, where a_i are non-zero constants and all

of the poles z_i are distinct. In that case, $\text{Res } g(z_i) = a_i$. Q.E.D.]

In the present example, this means

$$A = \text{Res}g(1) = \frac{1}{1-2} = -1$$

$$B = \text{Res}g(2) = \frac{1}{2-1} = 1.$$

You may wish to check using the usual method that these values are in fact correct.

Next, we divide both sides by one of the left-off powers in the denominator to get a function intermediate between $f(z)$ and $g(z)$,

$$h(z) = \frac{g(z)}{z-2} = \frac{-1}{(z-1)(z-2)} + \frac{1}{(z-2)^2}.$$

Now decompose the denominator of the first term on the right using exactly the same values of A and B as above,

$$\therefore h(z) = -1 \left[\frac{-1}{z-1} + \frac{1}{z-2} \right] + \frac{1}{(z-2)^2}.$$

Finish off the problem by dividing this by the remaining left-off power $(z-1)$, recognizing the second term as $g(z)$ and the third term as $h(z)$, then multiplying the final result by $3z$ to get

$$f(z) = \frac{3z}{(z-1)^2} + \frac{6z}{z-1} - \frac{6z}{z-2} + \frac{3z}{(z-2)^2}.$$

You can easily check this result by bringing all of the terms to the required common denominator.

Unlike the usual method, no solving of simultaneous equations is needed!