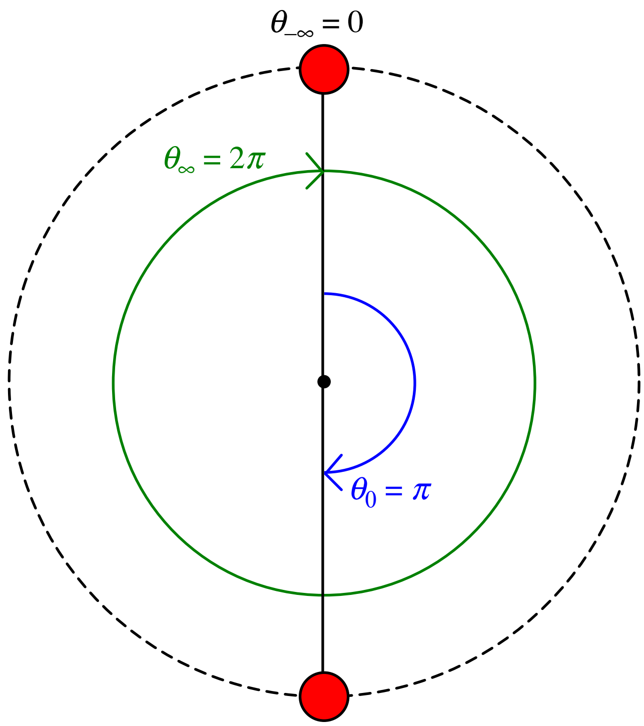
Pendulum Separatrix—C.E. Mungan, Summer 2021

<u>reference</u> *EJP 33:1555 (2012)*

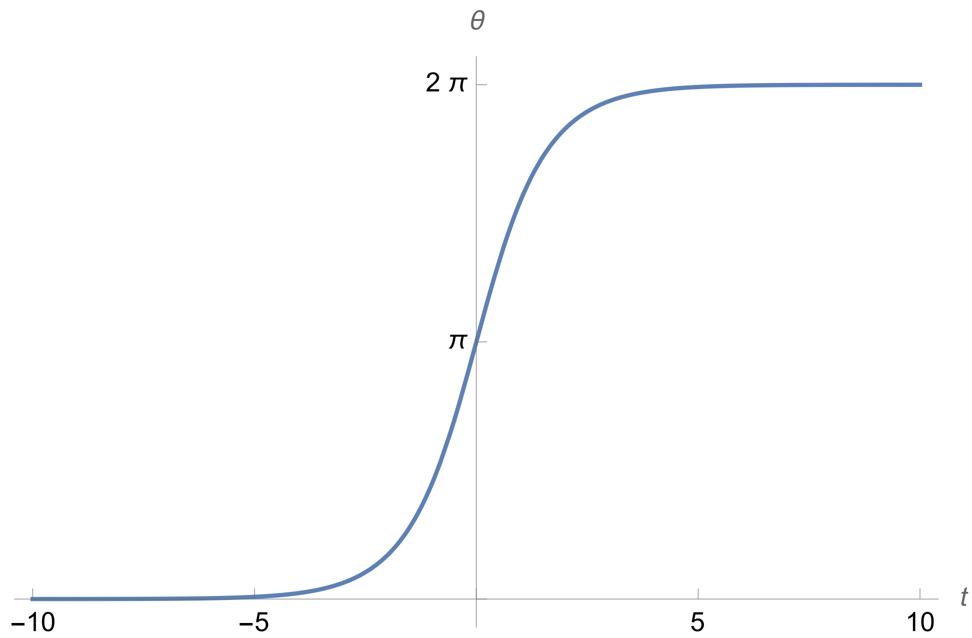
A simple pendulum (namely a point mass at the end of a massless rod of length L on a frictionless pivot which swings without drag) passes through its lowest point $\theta_0 = \pi$ at $t = 0$ and just makes it (i.e., as $t \to \infty$) around to the highest point $\theta_\infty = 2\pi$ of its circular arc. Find a formula for $\theta(t)$ in terms of elementary functions. For simplicity choose units such that the small-amplitude angular frequency $\omega = (g/L)^{1/2}$ is equal to 1.



The marvelous answer is

$$\boxed{\theta(t) = 4\tan^{-1} e^t} \tag{1}$$

or more generally replace *t* by ωt if ω is not equal to 1. A graph of Eq. (1) is presented at the top of the next page over all time *t*, which shows that θ has the correct values at times of $-\infty$, 0, and ∞. In other words, the bob starts in the distant past (i.e., as $t \to -\infty$) at the highest point $\theta_{-\infty} = 0$. Equation (1) describes the "separatrix" which separates pendulum from loop-the-loop motion.



To show that Eq. (1) is correct at all times, we must show that it satisfies conservation of mechanical energy in the form

$$\frac{d\theta}{dt} = \sqrt{2(1-\cos\theta)}\,. \tag{2}$$

Do that as follows. The time derivative of Eq. (1) is

$$\frac{d\theta}{dt} = \frac{4e^t}{1+e^{2t}} = \frac{4}{e^t+e^{-t}}\,. \tag{3}$$

However, Eq. (1) also implies that

$$\tan\frac{\theta}{4} = e^t \quad \Rightarrow \quad \cos\frac{\theta}{4} = \frac{1}{\sqrt{1+e^{2t}}}\,. \tag{4}$$

But the double-angle formula for cosine implies

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} \tag{5}$$

so that

$$\cos\frac{\theta}{4} = \sqrt{\frac{1+\cos(\theta/2)}{2}}\,. \tag{6}$$

Equate the right-hand sides of Eqs. (4) and (6) to get

$$1+e^{2t} = \frac{2}{1+\cos(\theta/2)} \quad \Rightarrow \quad e^t = \sqrt{\frac{1-\cos(\theta/2)}{1+\cos(\theta/2)}}\,. \tag{7}$$

Substitute this result into Eq. (3) to obtain

$$\frac{d\theta}{dt} = \frac{4}{\sqrt{\frac{1-\cos(\theta/2)}{1+\cos(\theta/2)}} + \sqrt{\frac{1+\cos(\theta/2)}{1-\cos(\theta/2)}}}$$

$$= \frac{4}{\sqrt{\frac{1-\cos(\theta/2)}{1+\cos(\theta/2)} \cdot \frac{1-\cos(\theta/2)}{1-\cos(\theta/2)}} + \sqrt{\frac{1+\cos(\theta/2)}{1-\cos(\theta/2)} \cdot \frac{1+\cos(\theta/2)}{1+\cos(\theta/2)}}} \tag{8}$$

$$= 2\sin(\theta/2).$$

Finally, the double-angle formula for cosine can alternatively be written as

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} \tag{9}$$

so that

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}\ . \tag{10}$$

Substituting Eq. (10) into (8) leads to (2), as we wanted to show.

Appendix: Forward derivation of Eq. (1)

A complete forward solution (rather than the verification presented above) proceeds as follows. The bob has zero kinetic energy at the topmost point, and define the gravitational potential energy to also be zero there, so that the total mechanical energy of the bob is zero. The sum of the kinetic and potential energies when the bob has swung down to angle θ must then be

$$\tfrac{1}{2} mL^2\dot{\theta}^2 + mgL(-1+\cos\theta) = 0 \tag{A1}$$

in real units where the bob has mass m, the rod has length L, and the gravitational field has magnitude g. This expression rearranges into Eq. (2) in scaled units such that $(g/L)^{1/2} = 1$. Integrate Eq. (2) as

$$\int_{\pi}^{\theta} \frac{d\theta}{\sqrt{1-\cos\theta}} = \sqrt{2}\int_{0}^{t} dt \tag{A2}$$

where the two lower limits correspond to the instant the bob passes through the bottom point of its circular swing. Using the half-angle formula for sine, Eq. (A2) becomes

$$\int_{\pi}^{\theta} \frac{d\theta}{\sqrt{2}\sin(\theta/2)} = \sqrt{2}\int_{0}^{t} dt \quad \Rightarrow \quad \int_{\pi}^{\theta} \frac{(d\theta)/2}{\sin(\theta/2)} = t \quad \Rightarrow \quad \int_{\pi/2}^{\theta/2} \frac{du}{\sin u} = t \tag{A3}$$

where $u \equiv \theta/2$. Next use the double-angle formula for sine to get

$$t = \int_{\pi/2}^{\theta/2} \frac{du}{2\sin(u/2)\cos(u/2)} = \int_{\pi/2}^{\theta/2} \frac{\sec^2(u/2)}{2\tan(u/2)} du = \ln\tan(u/2)\Big|_{\pi/2}^{\theta/2} = \ln\tan\frac{\theta}{4} \tag{A4}$$

which inverts to become Eq. (1).