Completely Inelastic Collisions—C.E. Mungan, Fall 1998

Problem: Prove that a maximum amount of kinetic energy is lost in a completely inelastic collision between two point masses, as claimed on page 201 of Cutnell & Johnson for instance.

This problem can be solved by jumping into the center-of-mass frame, as is done in Engineering Mechanics. But the following approach is intended to be accessible to introductory physics students, for use in a Take-Home Exam for example by giving suitable hints outlining this solution. Let us first restrict ourselves to a 1D collision and avoid calculus, as appropriate for General Physics. The loss in kinetic energy can be written as

$$KE_{lost} = KE_i - KE_f = \tfrac{1}{2} m_1 \upsilon_{1i}^2 + \tfrac{1}{2} m_2 \upsilon_{2i}^2 - \tfrac{1}{2} m_1 \upsilon_{1f}^2 - \tfrac{1}{2} m_2 \upsilon_{2f}^2 . \tag{1}$$

But by conservation of momentum,

$$p_i = p_f \Rightarrow m_1 \upsilon_{1i} + m_2 \upsilon_{2i} = m_1 \upsilon_{1f} + m_2 \upsilon_{2f} . \tag{2}$$

Solve Eq. (2) for υ_{2f}, square that, and substitute it into Eq. (1) to obtain a quadratic equation in υ_{1f}, which can be written as

$$KE_{lost} = -A\upsilon_{1f}^2 + B\upsilon_{1f} - C, \tag{3}$$

where

$$A = \frac{m_1}{2m_2}(m_1 + m_2), \quad B = \frac{m_1}{m_2}(m_1\upsilon_{1i} + m_2\upsilon_{2i}), \quad \text{and}$$
$$C = \frac{m_1}{2m_2}\upsilon_{1i}(m_1\upsilon_{1i} - m_2\upsilon_{1i} + 2m_2\upsilon_{2i}) \tag{4}$$

after some straightforward algebra. The student should now be invited to plot Eq. (3) and convince herself that it describes an inverted parabola which consequently has a single well-defined maximum. However, in my experience even our physics seniors are not aware of the fact that any quadratic describes a parabola and they instead believe that Eq. (3) is parabolic only if $B = 0$. To convince both freshman and senior alike, one should complete the square (borrowing from the derivation of the quadratic equation) to rewrite (3) as

$$KE_{lost} = -A\left(\upsilon_{1f} - \frac{B}{2A}\right)^2 + \left(\frac{B^2}{4A} - C\right). \tag{5}$$

It is now plainly obvious to all that this is parabolic with its vertex at

$$\upsilon_{1f} = \frac{B}{2A} = \frac{m_1\upsilon_{1i} + m_2\upsilon_{2i}}{m_1 + m_2} \Rightarrow m_1\upsilon_{1i} + m_2\upsilon_{2i} = m_1\upsilon_{1f} + m_2\upsilon_{1f} \tag{6}$$

using Eq. (4). Comparison of this with Eq. (2) proves that $\upsilon_{2f} = \upsilon_{1f}$, which is the definition of a completely inelastic collision, when KE_{lost} is a maximum.

Let us now modify the above solution to handle the fully general case of a 3D collision. I will not avoid calculus this time, since the extra algebra is already too much for a General Physics student and would be more appropriate in a University Physics setting. Components are introduced by expanding $\upsilon_{1f}^2 = \upsilon_{1f,x}^2 + \upsilon_{1f,y}^2 + \upsilon_{1f,z}^2$ and likewise for υ_{2f}^2 in Eq. (1), with a rather unfortunate profusion of subscripts and superscripts, and rewriting Eq. (2) as three separate equations, one in each component. Each of these new versions of Eq. (2) is solved for the appropriate component of υ_{2f} and substituted into Eq. (1) to obtain a single equation for KE_{lost} in terms of the three unknown components of υ_{1f}. Finally, taking the partial derivative of this equation with respect to say $\upsilon_{1f,x}$ and setting the result equal to zero proves that $\upsilon_{2f,x} = \upsilon_{1f,x}$. Similarly the other two components of υ_{1f} and υ_{2f} are shown to be equal, so that the collision is again seen to be completely inelastic.