

Perpendicular-Axis Theorem for Volumes—C.E. Mungan, Spring 2005

Reference: *AJP* 57:181 (1989)

The perpendicular-axis theorem for a thin flat plate is well known to be

$$I_x + I_y = I_z, \quad (1)$$

where (x, y, z) are three mutually orthogonal axes with x and y in the plane of the lamina and z normal to it. A generalization of this theorem to three-dimensional objects is

$$\boxed{I_x + I_y + I_z = 2 \int r^2 dm}, \quad (2)$$

where (x, y, z) are any set of three orthogonal axes and r is the spherical radial distance of mass element dm from the origin (not to be confused with the cylindrical radius typically used in moment-of-inertia calculations).

The proof of Eq. (2) follows from the definition of the moments,

$$I_x + I_y + I_z = \int (y^2 + z^2) dm + \int (x^2 + z^2) dm + \int (x^2 + y^2) dm, \quad (3)$$

which simplifies to Eq. (2) since $r^2 = x^2 + y^2 + z^2$. For the special case of a lamina in the xy -plane, $z = 0$ so that the right-hand side of Eq. (3) becomes $2 \int (x^2 + y^2) dm = 2I_z$ and Eq. (1) immediately follows.

Equation (2) has two useful applications. First and most obviously, it provides a quick way to compute the moment of inertia of a spherically symmetric object. In this case, Eq. (2) reduces to

$$3I = 2 \int r^2 dm \quad (4)$$

provided the origin of the axes is chosen to be at the center of the object. For example, for a uniform spherical shell, the right-hand side is trivially $2R^2M$ so that we obtain the familiar result,

$$I(\text{spherical shell}) = \frac{2}{3} MR^2. \quad (5)$$

One should note that no calculus is required to obtain this result (since the integral can be replaced by a summation of $r_i^2 m_i$), making this a good way to derive this moment in an introductory algebra-based physics course. As a second example, Eq. (4) for a uniform solid sphere becomes

$$I(\text{solid sphere}) = \frac{2}{3} \int r^2 dm = \frac{2M}{R^3} \int_0^R r^4 dr = \frac{2}{5} MR^2 \quad (6)$$

since

$$dm = \rho dV = \frac{M}{V} dV = \frac{M}{\frac{4}{3}\pi R^3} 4\pi r^2 dr = \frac{3M}{R^3} r^2 dr. \quad (7)$$

The other application of Eq. (2) follows from the fact that it implies that the sum of the moments of inertia (i.e., the trace of the inertia tensor) is invariant under a rotation of the coordinate axes. This eliminates at least one recalculation of the moments for a rotated set of axes. In fact, in special cases it can eliminate any necessity to reintegrate at all. For example, consider again a lamina in the xy -plane. Suppose we have integrated I_x and I_y from their definitions, and summed them to find I_z from Eq. (1). Now suppose we wish to find the moments about the axes (x', y', z') obtained by rotating the z -axis 45° into the direction of the y -axis. By symmetry $I'_y = I'_z \equiv I'$ so that

$$\begin{aligned} I_x + I_y + I_z &= I'_x + I'_y + I'_z \\ \Rightarrow 2(I_x + I_y) &= I_x + 2I' \\ \Rightarrow I' &= \frac{1}{2}I_x + I_y. \end{aligned} \quad (8)$$

For example, for a uniform circular hoop (for which $I_x = I_y = \frac{1}{2}I_z = \frac{1}{2}MR^2$), this equation implies that $I' = \frac{3}{4}MR^2$, midway in value between I_y and I_z .