

Phase Change upon Reflection—C.E. Mungan, Spring 2008

It is straightforward to derive the rule for phase changes upon reflection of a plane electromagnetic wave at normal incidence to the interface between two dielectric media. The derivation uses the following two propositions which are proven in Appendix I. These propositions can be taken as plausible without formally verifying them in an introductory course. First, the tangential component of the total electric field is continuous across the interface,

$$E_{1,\text{tan}} = E_{2,\text{tan}} \quad (1)$$

where the incident medium is denoted by the subscript “1” and the medium into which the wave is transmitted as “2.” Second, the tangential component of the total magnetic field is also continuous across the interface,

$$B_{1,\text{tan}} = B_{2,\text{tan}} \quad (2)$$

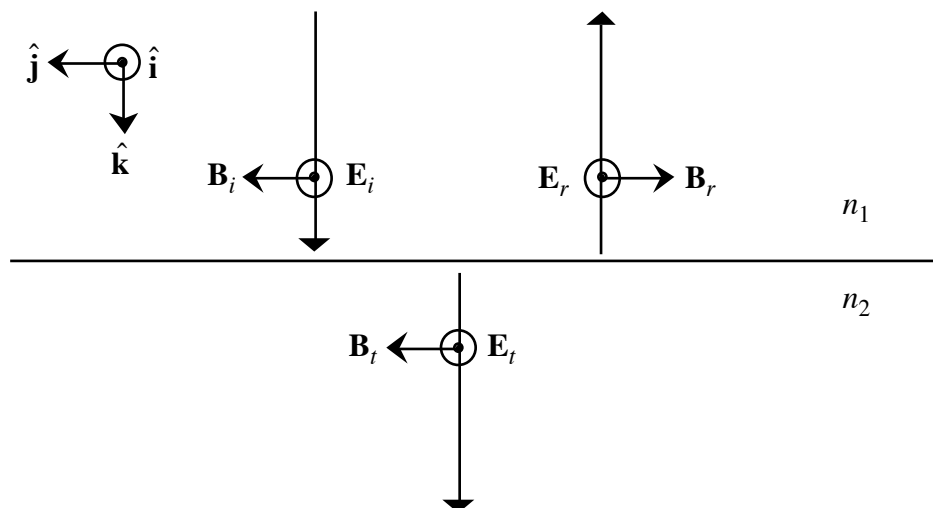
Standard introductory textbooks use Maxwell’s equations to prove that the ratio of the electric to the magnetic field strengths at any position in space and instant in time can be rearranged as

$$B = \frac{E}{v} \quad (3)$$

for a plane wave, where the propagation speed of the wave in a nonmagnetic medium is related to the index of refraction by

$$v = \frac{c}{n} = \frac{1}{\sqrt{\epsilon\mu_0}} \quad (4)$$

according to the wave equation. In other words, the index of refraction equals the square root of the frequency-dependent dielectric constant $\kappa = \epsilon / \epsilon_0$.



The derivation proceeds as follows, referring to the above diagram. Without loss of generality, choose the z -axis to be parallel to the directions of propagation, let the direction of the incident electric field \mathbf{E}_i at the interface at some instant in time define the x -axis, and suppose that the reflected electric field \mathbf{E}_r and the transmitted electric field \mathbf{E}_t at the same instant also

point in the x direction, as sketched above. (Specifically the reflected and transmitted electric fields can point in either the positive or negative x directions, depending on the signs of E_r and E_t , respectively.) The directions of the magnetic fields \mathbf{B} have then been chosen in the diagram so that $\mathbf{E} \times \mathbf{B}$ is in the direction of propagation of the waves. Equation (1) implies

$$\sum E_{1x} = \sum E_{2x} \Rightarrow E_i + E_r = E_t, \quad (5)$$

while Eq. (2) becomes

$$\sum B_{1y} = \sum B_{2y} \Rightarrow B_i - B_r = B_t \Rightarrow \frac{E_i}{v_1} - \frac{E_r}{v_1} = \frac{E_t}{v_2} \quad (6)$$

after substituting Eq. (3). Next substitute Eq. (4) into Eq. (6) to get

$$n_1 E_i - n_1 E_r = n_2 E_t = n_2 E_i + n_2 E_r, \quad (7)$$

using Eq. (5) to obtain the second equality. Rearrange this result to obtain the amplitude reflection coefficient for normal incidence,

$$\boxed{\frac{E_r}{E_i} \equiv r = \frac{n_1 - n_2}{n_1 + n_2}}. \quad (8)$$

Consider the three possible relative values of the refractive indices:

- (a) $n_1 = n_2$ in which case $r = 0$, i.e., there is no reflected wave because of index matching;
- (b) $n_1 > n_2$ in which case \mathbf{E}_r is in the direction of \mathbf{E}_i , i.e., there is no phase change;
- (c) $n_1 < n_2$ in which case \mathbf{E}_r is in the opposite direction as \mathbf{E}_i , i.e., there is a 180° phase change.

A portable demonstration of index matching uses anisole in a borosilicate glass eyedropper bottle, as described in TPT **18**, 467 (Sep 1980).

Remarks

1. Note that Eq. (8) can be rewritten as

$$\frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2} = \frac{c/v_1 - c/v_2}{c/v_1 + c/v_2} = \frac{v_2 - v_1}{v_2 + v_1}. \quad (9)$$

Passing from a low-index to a high-index medium ($n_1 < n_2$) leads to a decrease in wavespeed ($v_2 < v_1$). According to case (c) above, the reflected wave undergoes a 180° phase change. This is analogous to a transverse wave pulse incident from a low-density to a high-density string, for which the reflected pulse is similarly inverted. Textbooks could improve their string/light wave analogy for phase reflection if they built their argument as follows:

- (i) A string wave is inverted when it reflects off a higher density medium.
- (ii) Since the incident medium has lower density, the wavespeed in it is higher.
- (iii) By analogy a light wave is inverted if the wavespeed is higher in the incident medium.
- (iv) In conclusion an electromagnetic wave is inverted if it reflects off a medium of higher index.

Nonetheless I think the derivation presented above is more convincing and yet quite accessible.

2. The intensity of a light wave is related to its root-mean-square electric field by

$$I = E_{rms}^2 / \mu_0 v. \quad (10)$$

Thus the intensity reflection coefficient (known as the reflectance) is

$$R \equiv \frac{I_r}{I_i} = r^2 = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 \quad (11)$$

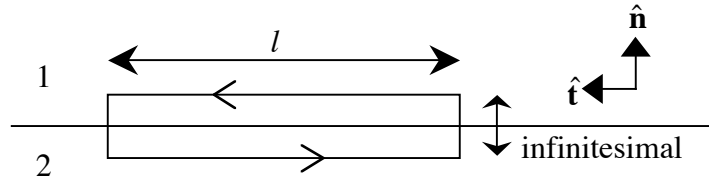
in agreement with the Fresnel equations. It is left as an exercise to the reader to verify using Eqs. (5) and (7) that the amplitude transmission coefficient for normal incidence is

$$\frac{E_t}{E_i} \equiv t = \frac{2n_1}{n_1 + n_2}, \quad (12)$$

so that there is never a phase change in transmission. Note that t can get as large as 2 (in contrast to r which cannot exceed 1) like a pulse reaching a free end of a string and flipping it through a distance equal to twice the amplitude of the pulse.

Furthermore $R + T = 1$, in accord with energy conservation, where the transmittance is $T \equiv I_t / I_i = t^2 n_2 / n_1$ for normal incidence. Note that T and R are symmetric in n_1 and n_2 . That is, the transmittance and reflectance are the same regardless of whether the beam is incident from the high or low index side of the interface. This conclusion can be neatly related to the Stokes relations based on the principle of optical reversibility, as discussed in Appendix II. Alternatively one can derive Eq. (8) using $I_i = I_r + I_t$ and Eq. (1), introduced in the context of the polarization of an electromagnetic wave, without appealing to Eq. (2) which may be less obvious to students.

Appendix I



Faraday's law applied to the loop sketched above of length l and infinitesimal height straddling the interface between the two media is

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A} \Rightarrow (\mathbf{E}_1 - \mathbf{E}_2) \cdot \hat{\mathbf{t}} = 0 \quad (13)$$

where $\hat{\mathbf{t}}$ is any unit vector tangential to the interface. The right-hand side is zero since the area of the loop tends to zero while the time derivative of \mathbf{B} is finite. Equation (13) is equivalent to Eq. (1).

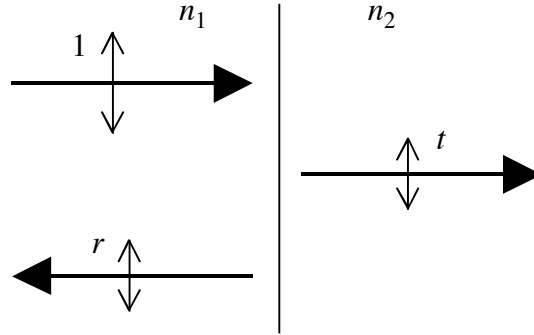
Ampère's generalized law for nonmagnetic media applied to this same loop is

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \int \mathbf{J} \cdot d\mathbf{A} + v^{-2} \frac{d}{dt} \int \mathbf{E} \cdot d\mathbf{A} \Rightarrow (\mathbf{B}_1 - \mathbf{B}_2) \cdot \hat{\mathbf{t}} = \mu_0 \mathbf{K} \cdot l(\hat{\mathbf{n}} \times \hat{\mathbf{t}}) \quad (14)$$

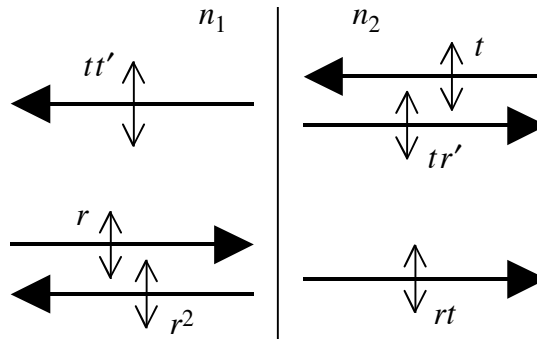
where $\hat{\mathbf{n}}$ is the unit normal vector and \mathbf{K} is the surface current density on the interface. Assuming nonconducting media, the right-hand side of Eq. (14) is zero, which consequently is equivalent to Eq. (2).

Appendix II

The following treatment is based on Hecht *Optics* Sec 4.10. Suppose a beam of unit transverse amplitude is incident from medium 1 onto medium 2. A fraction r of it is reflected and a fraction t is transmitted as sketched below.



Now reverse the two outgoing rays and consider the reflected and transmitted portion of each of them as drawn below. Denote r' and t' as the respective reflection and transmission coefficients for incidence from medium 2 onto medium 1.



The principle of optical reversibility says that the superposed outgoing amplitudes on each side of the interface here must agree with those of the corresponding incoming rays in the top figure. We therefore conclude, by summing the outgoing rays in medium 2 in the bottom figure, that

$$tr' + rt = 0 \Rightarrow \boxed{r' = -r}, \quad (15)$$

consistent with the sign reversal of Eq. (8) if one exchanges n_1 and n_2 . It follows that $R' = R$ using Eq. (11). Since $R' = 1 - T'$ and $R = 1 - T$, it also follows that $T' = T$. These two results imply that the reflectance and transmittance are the same for incidence from either medium.

We also conclude, by summing the outgoing rays in medium 1 in the bottom figure, that

$$tt' + r^2 = 1 \Rightarrow tt' = 1 - R \Rightarrow \boxed{tt' = T}. \quad (16)$$

Substitute Eq. (12), and the analogous equation for t' when we exchange n_1 and n_2 , to obtain

$$tt' = \frac{2n_1}{n_1 + n_2} \cdot \frac{2n_2}{n_1 + n_2} \Rightarrow T = \frac{4n_1n_2}{(n_1 + n_2)^2} \quad (17)$$

which agrees with the result $T = t^2 n_2 / n_1$ following Eq. (12).