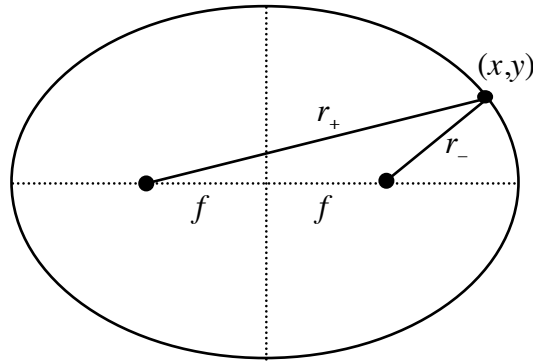


## Polar Form of an Ellipse—C.E. Mungan, Summer 2015

In this document, I derive three useful results: the polar form of an ellipse, the relation between the semilatus rectum and the angular momentum, and a proof that an ellipse can be drawn using a string looped around the two foci and a pencil that traces out an arc. The ingredients are the rectangular form of an ellipse, the conserved angular momentum and mechanical energy, and definitions of various elliptical parameters.



Let's start with the looped string property. Putting the origin at the geometric center in the sketch above, the rectangular form of an ellipse is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \tag{1}$$

where the semi-major axis is  $a$  and the semi-minor axis is  $b$ . Define the focal length as

$$f \equiv \sqrt{a^2 - b^2} . \tag{2}$$

Combining Eqs. (1) and (2), we get

$$y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right) = (a^2 - f^2) \left(1 - \frac{x^2}{a^2}\right) = a^2 - f^2 - x^2 + \frac{f^2}{a^2} x^2 . \tag{3}$$

However, applying the Pythagoras relation in the sketch above, we see that

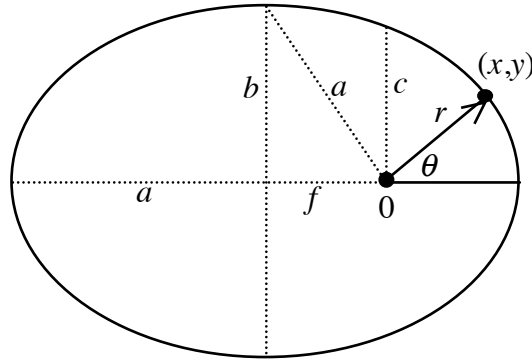
$$r_{\pm}^2 = (x \pm f)^2 + y^2 = \left(a \pm \frac{f}{a} x\right)^2 \tag{4}$$

using Eq. (3) in the second step. We conclude that

$$\begin{aligned}
r_+ &= a + \frac{f}{a}x \\
r_- &= a - \frac{f}{a}x
\end{aligned}
\Rightarrow \boxed{r_+ + r_- = 2a} \tag{5}$$

which proves the property that the sum of the distances from any arbitrary point on the ellipse to the two foci is a constant. By choosing the arbitrary point to be one of the two apses of the ellipse, it becomes apparent that that constant must be  $2a$ . That in turn implies that the diagonal dotted line in the next diagram has length  $a$ , consistent with Eq. (2).

Next we'll go on to derive the polar form of an ellipse. For this purpose, it is convenient to shift the origin to one of the two foci, as indicated below.



Consequently we need to add  $f$  to this new  $x$  to get what was called  $x$  in Eq. (1),

$$\left(\frac{x+f}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \tag{6}$$

Multiply this equation through by  $(ab)^2$  and substitute

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta \tag{7}$$

to get

$$b^2 r^2 \cos^2 \theta + 2b^2 f r \cos \theta + b^2 f^2 + a^2 r^2 \sin^2 \theta = a^2 b^2. \tag{8}$$

Use Pythagoras to replace  $\sin^2 \theta = 1 - \cos^2 \theta$  and obtain

$$a^2 r^2 + 2b^2 f r \cos \theta - (a^2 - b^2) r^2 \cos^2 \theta = b^2 (a^2 - f^2). \tag{9}$$

Now replace both quantities in parentheses using Eq. (2) to get

$$a^2 r^2 + 2b^2 f r \cos \theta - f^2 r^2 \cos^2 \theta = b^4 \Rightarrow (ar)^2 = (b^2 - fr \cos \theta)^2. \tag{10}$$

Take the (positive) square root of both sides and solve for  $r$  to obtain

$$r = \frac{b^2/a}{1 + e \cos \theta} \quad (11)$$

where the eccentricity is defined as  $e \equiv f/a$ . Finally, introduce the semilatus rectum  $c$  drawn in the second sketch above. We see from that diagram that  $r = c$  when  $\theta = 90^\circ$ , in which case Eq. (11) implies

$$c = b^2/a. \quad (12)$$

[This result can also be obtained directly from the rectangular form in Eq. (6) by substituting  $x = 0$  and  $y = c$  along with Eq. (2).] Hence Eq. (11) can be written in the conventional polar form

$$\boxed{r = \frac{c}{1 + e \cos \theta}}. \quad (13)$$

Last of all, we introduce the two physically conserved quantities. At periapsis, the specific angular momentum of the satellite of mass  $m$  and speed  $v$  is

$$h \equiv \frac{L}{m} = r_p v_p \Rightarrow v_p = \frac{h}{r_p} \quad (14)$$

and likewise at apoapsis,

$$v_a = \frac{h}{r_a}. \quad (15)$$

If the satellite is orbiting about a body of mass  $M$  (which is much larger than  $m$  so that we can neglect the motion of  $M$ ) then the mechanical energy at the two apses is

$$E = \frac{1}{2} m v_p^2 - \frac{GMm}{r_p} = \frac{1}{2} m v_a^2 - \frac{GMm}{r_a} \Rightarrow \frac{h^2}{r_p^2} - \frac{2GM}{r_p} = \frac{h^2}{r_a^2} - \frac{2GM}{r_a} \quad (16)$$

using Eqs. (14) and (15). Next note that

$$\frac{1}{r_p} - \frac{1}{r_a} = \frac{1}{a-f} - \frac{1}{a+f} = \frac{2f}{a^2 - f^2} = \frac{2f}{b^2} \quad (17)$$

using Eq. (2) in the last step. Likewise,

$$\frac{1}{r_p^2} - \frac{1}{r_a^2} = \frac{4af}{b^4}. \quad (18)$$

Substituting these two differences of reciprocals into Eq. (16) leads to

$$h^2 \frac{4af}{b^4} = 2GM \frac{2f}{b^2} \quad (19)$$

and thus

$$\boxed{\frac{h^2}{GM} = \frac{b^2}{a} = c} \quad (20)$$

using Eq. (12) in the last step. This result usefully relates the semilatus rectum to the angular momentum and masses.