

Relationship between Displacement and Pressure Amplitude—C.E. Mungan, Fall 2000

Consider a monochromatic plane sound wave traveling down the length of a tube of gas of ambient density ρ_0 at a phase speed v_s . Find an expression for the relationship between the amplitude of the molecular displacement and that of the pressure oscillations.

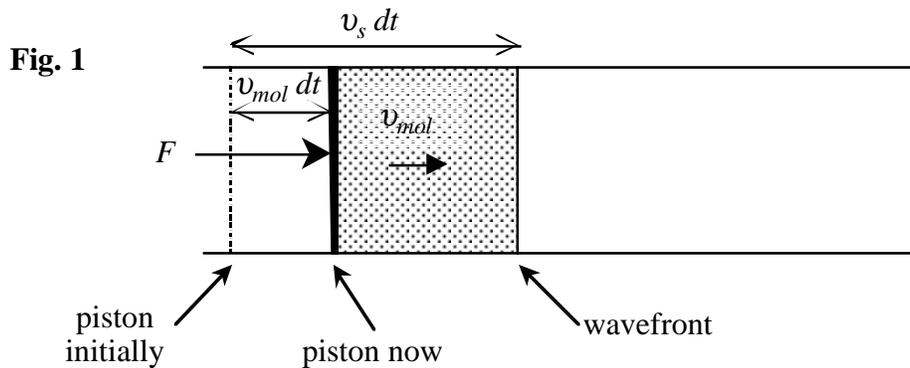
As is proven below in three different ways, the answer is

$$P = \rho_0 v_s v_{mol} \quad (1)$$

where P is the gauge pressure and v_{mol} is the longitudinal molecular speed due to the sound wave (i.e., averaged over all molecules within a thin transverse slice so that their random thermal motions can be neglected). Since the maximum longitudinal speed is equal to $A\omega$ where A is the displacement amplitude of the molecules from their (average) equilibrium position and ω is the angular frequency of the sound, we conclude that the pressure amplitude is

$$P_{max} = \rho_0 v_s \omega A, \quad (2)$$

in agreement with Serway Eq. 13.24.



Proof #1 (Marion and Hornyak): Suppose the sound wave is generated by a piston which fits snugly in the left end of a long tube of cross-sectional area S , which is open at the right end to the ambient surroundings, as sketched in Fig. 1 above. Initially, let the piston be in static equilibrium. Now consider applying a rightward force F to the piston. This will generate a compression whose wavefront will move some distance dx to the right of the original position of the piston during a small interval of time dt . Since this wavefront moves at the speed of sound, it follows that $dx = v_s dt$. The molecules themselves, however, move at the lower speed v_{mol} imparted by the piston. Since the distance from the piston to the wavefront is differentially small, we can assume that all of the molecules in this volume have the same speed. Clearly, the total mass of gas moving to the right is $dm = \rho_0 S dx$, with a corresponding linear momentum of

$$dp = \rho_0 S v_s dt v_{mol}. \quad (3)$$

But by Newton's second law, this must result from a pressure in excess of ambient of

$$P \equiv \frac{F}{S} = \frac{1}{S} \frac{dp}{dt} = \rho_0 v_s v_{mol}. \quad (4)$$

Proof #2: This is merely a reworking of the first proof but is instructive for students who have been introduced to the definition of the adiabatic bulk modulus B ,

$$\Delta P = -B \frac{\Delta V}{V}. \quad (5)$$

The original volume of the gas which is dotted in Fig. 1 is $V = Sv_s dt$, and the change in this volume due to the compression by the piston is $\Delta V = -Sv_{mol} dt$. The increase in pressure on the piston necessary to accomplish this is just the gauge pressure, $\Delta P = P$. Finally, the speed of sound is well-known to be given by

$$v_s = \sqrt{\frac{B}{\rho_0}} \Rightarrow B = \rho_0 v_s^2. \quad (6)$$

This can be derived from the wave equation, for example. Substituting these expressions for V , ΔV , ΔP , and B into Eq. (5) gives Eq. (1).

Proof #3 (Resnick, Halliday, and Krane): Consider one compressional pulse out of a square wave traveling with speed v_s down a tube from left to right. View it in the rest frame of the pulse, as sketched in Fig. 2 below. (This is the same frame that was used to find the speed of a transverse wave on a string in Serway Sec. 13.6.) In this frame, the region of compression is stationary, while fluid moves through it from right to left. Consider a slice of fluid of thickness dx moving toward the compressed region at speed v_s . The right edge of the slice enters the compression a time dt after the left edge does, where $dx = v_s dt$. During the time interval dt while the slice is entering the compression, the pressure on the leading face of the fluid element exceeds that on the trailing face by the gauge pressure P . Consequently the leading edge of the slice is decelerated, as it must be in order to get compressed. Inside the compression, the fluid travels at the lower speed of $v_s - v_{mol}$, since the molecules inside the pulse are traveling to the right at a speed of v_{mol} in the lab frame. We now apply Newton's second law to find

$$F = ma \Rightarrow PS = (\rho_0 S v_s dt) \frac{v_s - (v_s - v_{mol})}{dt}, \quad (7)$$

which is Eq. (1) when simplified.

