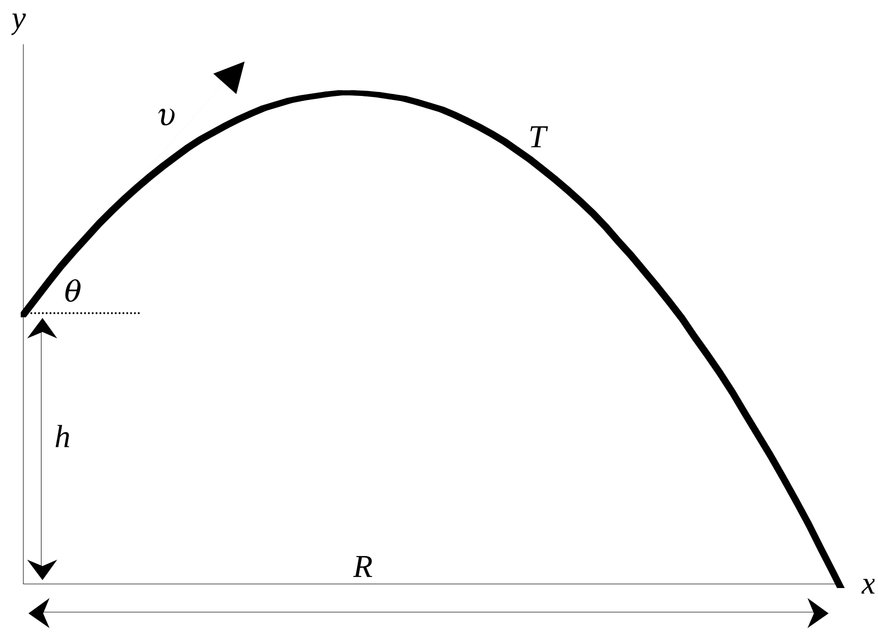
Maximum Range of a Projectile Launched from a Height—C.E. Mungan, Spring 2003

<u>*reference*</u>*: TPT 41:132 (March 2003)*

Find the launch angle θ and maximum range R of a projectile launched from height h at speed υ.



The basic equations of kinematics at the landing point after flight time T are

$$0 = h + \upsilon_y T - \tfrac{1}{2} g T^2 \tag{1}$$

vertically and

$$R = \upsilon_x T \tag{2}$$

horizontally. Substitute Eq. (2) for T into (1) and convert from rectangular to polar components to get

$$h(1 + \cos 2\theta) = \frac{gR^2}{\upsilon^2} - R \sin 2\theta . \tag{3}$$

Maximize R by differentiating this expression with respect to θ and putting $dR / d\theta = 0$ to obtain an expression for the optimum launch angle,

$$\tan 2\theta = \frac{R}{h} \quad \Rightarrow \quad \theta = \frac{1}{2} \tan^{-1} \frac{R}{h} . \tag{4}$$

This implies $\cos 2\theta = h(h^2 + R^2)^{-1/2}$ and $\sin 2\theta = R(h^2 + R^2)^{-1/2}$. Substitute these into Eq. (3) to obtain the maximum range,

$$h = \frac{gR^2}{2\upsilon^2} - \frac{\upsilon^2}{2g} \quad \Rightarrow \quad R = \sqrt{\frac{2\upsilon^2}{g}\left(h + \frac{\upsilon^2}{2g}\right)} . \tag{5}$$

Equations (4) and (5) can be normalized for plotting purposes in terms of

$$R_0 \equiv \frac{v^2}{g}, \tag{6}$$

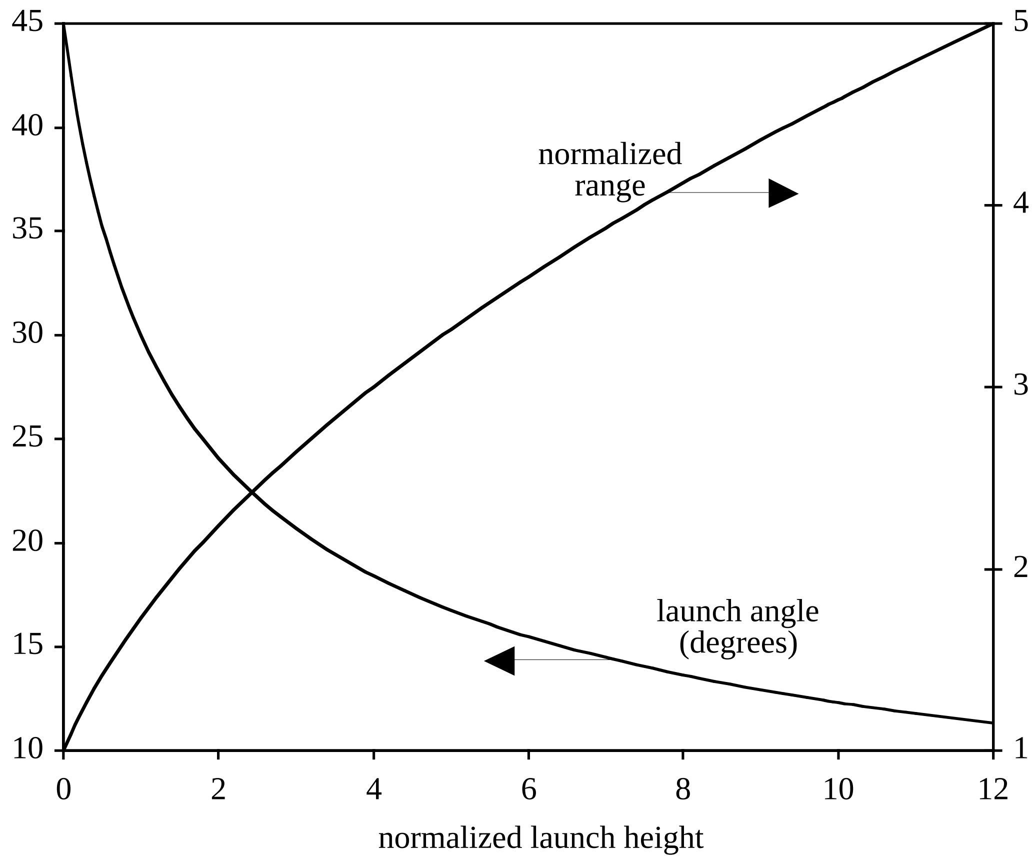
the maximum range for a surface-to-surface projectile (i.e., when $h = 0$), to get the normalized range

$$\frac{R}{R_0} = \sqrt{1 + 2\frac{h}{R_0}} \tag{7}$$

at a launch angle of

$$\theta = \frac{1}{2}\sec^{-1}\left(1 + \frac{R_0}{h}\right). \tag{8}$$

These two equations are plotted below as a function of the normalized launch height h / R_0.



As expected, $R = R_0$ and $\theta = 45^\circ$ when $h = 0$. In the other limit, $R \propto h^{1/2}$ and $\theta \to 0^\circ$ as $h \to \infty$. More reasonably, notice that if you launch from a height equal to 1.5 times your surface range, you can get the projectile to go twice as far, provided you launch it at 26.6˚ (half of a 3–4–5 triangle angle).