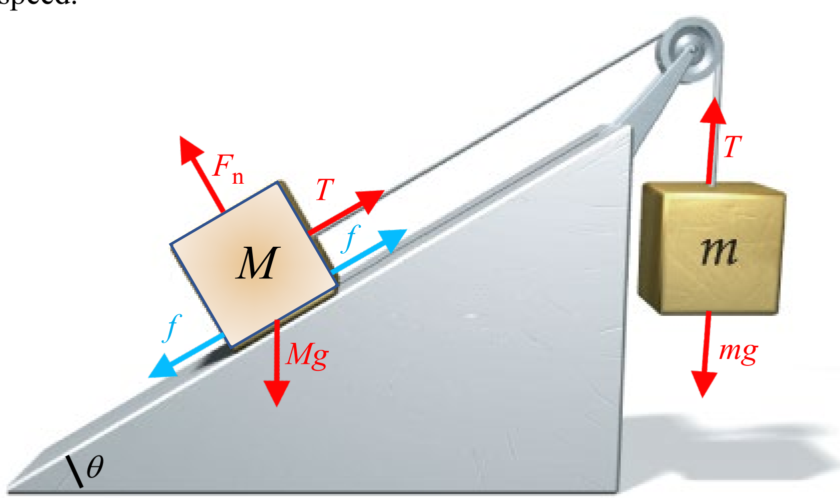
Tipler & Mosca Problem 5.62—C.E. Mungan, Fall 2023

A block of mass M on an incline is connected to a hanging weight of mass m via an ideal string and pulley as shown below. The coefficients of static and kinetic friction between M and the incline are μ_s and μ_k respectively. The ramp is inclined at angle θ with respect to the horizontal.

(a) Determine the range of values of m for the which the block on the incline will not move if undisturbed, but if the block is given a downward push it will slide down the incline without decelerating in speed.

(b) Determine the range of values of m for the which the block on the incline will not move if undisturbed, but if the block is given an upward push it will slide up the incline without decelerating in speed.



Solution: The forces have been labeled with red and blue arrows on the above diagram. Friction f can be either static or kinetic, and can point either up or down the incline, as discussed below. The normal force on the block is always $F_n = Mg\cos\theta$ since the block does not accelerate perpendicular to the incline. Furthermore, we will only be interested in cases where the system is at rest or where the system moves with constant speed. Either way, the acceleration of the masses is therefore zero so that the tension in the string will always be $T = mg$.

Let's start by considering what happens when the system does not move so that the friction is static. Suppose m is large enough that M is about to slip up the incline, so that f points down the incline and is at its maximum static value. Then the sum of the force components up the incline is

$$T - f_{s\,max} - Mg\sin\theta = 0 \quad \Rightarrow \quad m = M(\sin\theta + \mu_s\cos\theta) \tag{1}$$

using $f_{s\,max} = \mu_s F_n$. This is the maximum value of m so that the block at rest remains at rest. Now suppose we instead make m small enough that M is about to slip down the incline. In that case we reverse the direction of static friction, which we can accomplish by changing the sign in front of μ_s in Eq. (1) to get

$$m = M(\sin\theta - \mu_s\cos\theta) \tag{2}$$

assuming $\tan\theta \ge \mu_s$ so that the answer is not negative. This is the minimum value of m so that the block at rest remains at rest. If $\tan\theta < \mu_s$, then m can be as small as zero, which means the block will remain at rest on the incline even if we eliminate the hanging block and string.

Next suppose the block is given a nudge down the incline. Then the friction *f* is kinetic and points up the incline. The block will keep moving down the incline (rather than slowing to a stop) if *m* is not too large. We can find the limiting value when the downward acceleration of the block is zero. Then the force components down the incline sum up to

$$Mg\sin\theta - T - f_k = 0 \quad \Rightarrow \quad m = M(\sin\theta - \mu_k\cos\theta) \tag{3}$$

using $f_k = \mu_k F_n$ and assuming $\tan\theta \geq \mu_k$ to ensure the answer is not negative. This is the maximum value of *m* so that the block continues to slide down the incline. If $\tan\theta < \mu_k$ then the block will decelerate to a stop for any value of *m*, no matter how small.

Finally suppose the block is given a nudge up the incline. It will keep moving up the incline if *m* is large enough. We again find the limiting value when the upward acceleration of the block is zero. We need to reverse the direction of kinetic friction by changing the sign in front of μ_k in Eq. (3) to get

$$m = M(\sin\theta + \mu_k\cos\theta)\,. \tag{4}$$

This is the minimum value of *m* so that the block continues to slide up the incline.

For example, suppose $M = 100$ kg, $\mu_s = 0.4$, $\mu_k = 0.2$, and $\theta = 30°$. Then the solution to part (a) is $15.4\text{ kg} \leq m \leq 32.7\text{ kg}$, and the solution to part (b) is $67.3\text{ kg} \leq m \leq 84.6\text{ kg}$. To summarize, these four limiting values are given by calculating $M(\sin\theta \pm \mu\cos\theta)$ where $\mu = \mu_k$ or μ_s. As a second example, if $M = 100$ kg, $\mu_s = 0.4$, $\mu_k = 0.2$, and $\theta = 20°$ then the solution to part (a) is $0 \leq m \leq 15.4$ kg, and the solution to part (b) is $53.0\text{ kg} \leq m \leq 71.8\text{ kg}$. Finally a third example is $M = 100$ kg, $\mu_s = 0.4$, $\mu_k = 0.2$, and $\theta = 10°$. Then there are no possible values of *m* that satisfy part (a), but the solution to part (b) is $37.1\text{ kg} \leq m \leq 56.8\text{ kg}$. The problem always has a solution for part (b) with two nonzero values of *m* (assuming $\mu_k < \mu_s$ as usual) and so it would be simpler to restrict the problem to asking for that case of upward sliding only.