

Radiative Coupling Between an Object and its Surroundings—C.E. Mungan, Spring 1999

The following situation arises in laser cooling and many other contexts. A sample (denoted here by a subscript “s”) is suspended inside an enclosed, constant-temperature cavity (denoted by a subscript “c”). The suspension is assumed to be by vanishingly thin, low thermal conductivity supports and the cavity is evacuated, so that there is no significant conductive or convective heat transfer between the sample and the cavity. Furthermore, we will assume the facing surfaces of the sample and cavity are each made of an opaque, homogeneous graybody, so that each has a single well-defined surface area (A_s and A_c) and emissivity (ϵ_s and ϵ_c), and that the radiative heat transfer between the two is sufficiently slow that each always has a single well-defined temperature (T_s and T_c) which are not necessarily equal because the sample is being laser cooled.

What is the radiative heat load P_{load} on the sample from the cavity in watts? We cover this every year in General Physics, University Physics, and Optics. The well-known answer is given by, for example, Eq. (13.3) of Cutnell & Johnson,

$$P_{load} = \epsilon_s A_s \sigma (T_c^4 - T_s^4) \quad (1)$$

obtained by subtracting the sample’s emission from its absorption and using Kirchoff’s law, which states that the absorptivity α_s of the sample is equal to its emissivity,

$$\alpha_s = \epsilon_s. \quad (2)$$

All very straightforward, right? It was therefore with a shock that while reading a recent paper, I learned that Eq. (1) is wrong! The correct expression is

$$P_{load} = \frac{\sigma}{R} (T_c^4 - T_s^4) \quad (3)$$

where R is the surface thermal resistance given by

$$R = \frac{1 - \epsilon_s}{\epsilon_s A_s} + \frac{1}{A_s F_{sc}} + \frac{1 - \epsilon_c}{\epsilon_c A_c}. \quad (4)$$

Here F_{sc} is known as the configuration factor and is defined as the fraction of the radiosity leaving the sample and striking the cavity walls directly; it depends solely on geometrical quantities and is calculated for a variety of standard geometries in engineering handbooks such as Schaum’s Outline Series in Heat Transfer. Note that the radiosity J , in turn, is defined as the total (i.e., over all wavelengths and directions) radiant thermal energy leaving a surface per unit time per unit area of the surface (with units of W/m^2)—it is the sum of the emitted and reflected intensities. The corresponding total incident radiant power per unit area of the surface is called the irradiance G .

Although Eq. (4) appears asymmetric in the sample and cavity variables, it is not in fact, since

$$A_s F_{sc} = A_c F_{cs}, \quad (5)$$

which is known as the reciprocity theorem and is easily proved in Schaum’s. Now, for simplicity let’s assume that the sample and cavity consist of two concentric cylinders. It is then clear that $F_{sc} = 1$ because every ray leaving the sample surface must strike the cavity walls. (In contrast, $F_{cs} < 1$ because a ray leaving one point on the cavity walls can miss the sample and strike another point on the walls. The cavity “sees itself” while the sample does not.) In that case, Eq. (4) can be simplified and substituted into Eq. (3) to obtain the final result

$$P_{load} = \frac{\epsilon_s A_s \sigma}{1 + \epsilon_s A_s (1 - \epsilon_c) / \epsilon_c A_c} (T_c^4 - T_s^4). \quad (6)$$

This reduces to Eq. (1) only if $\epsilon_c = 1$ (i.e., the cavity walls are black and not merely opaque) or if

the emissivity or surface area of the sample is much smaller than that of the cavity. A perusal of introductory physics texts reveals no hint of these limitations in the use of Eq. (1) however.

Well perhaps that is because these books are too introductory, you might think. Surely a higher level book such as Hecht (3rd edition) would take into account the emissivity of the cavity walls. Sure enough, he does on page 577. Unfortunately he takes it into account in an egregiously incorrect manner. He balances the sample's emission $\epsilon_s A_s \sigma T_s^4$ against its absorption which he writes as $\alpha_s A_s \cdot \epsilon_c \sigma T_c^4$ under the equilibrium condition $T_s = T_c$ to conclude that

$$\alpha_s \epsilon_c = \epsilon_s \quad (7)$$

in violation of Kirchhoff's law, as expressed by Eq. (2). The fact that this result implies that the absorptivity and/or emissivity of the sample and cavity depend on each other and could thus not be intrinsic quantities is passed over in silence, despite the fact that he subsequently tabulates typical values of emissivity for various materials!

Clearly then, both our introductory and our intermediate level textbooks are confused on the topic of radiative heat transfer. So perhaps it is useful to review the derivation of Eq. (4). Schaum's uses a three-resistor analogy to obtain it. (As far as I can tell, if the holy grail of physics is the unification of the 3 fundamental forces, the highest goal of engineering is to reduce all practical problems to parallel and series electrical circuits.) The net heat load on the sample is given by

$$P_{load} / A_s = G_s - J_s \quad (8)$$

where, as already mentioned, the radiosity is the sum of the emitted and reflected terms,

$$J_s = \epsilon_s \sigma T_s^4 + (1 - \epsilon_s) G_s \quad (9)$$

using the fact that, by energy conservation, the sum of the reflectance and absorptance must be unity for an opaque sample, so that the reflectance equals $1 - \epsilon_s$. Eliminating G_s between Eqs. (8) and (9) and rearranging, we obtain

$$J_s = \sigma T_s^4 + P_{load} \frac{1 - \epsilon_s}{\epsilon_s A_s}. \quad (10)$$

Similarly, by shifting focus to the cavity and noting that the net heat gained by the sample must equal the net heat lost by the cavity walls, we have

$$J_c = \sigma T_c^4 - P_{load} \frac{1 - \epsilon_c}{\epsilon_c A_c}. \quad (11)$$

Finally, F_{sc} is the configuration factor relating the radiosity from the sample surface to that of the cavity walls,

$$P_{load} / A_s = F_{sc} (J_c - J_s). \quad (12)$$

For example, in the present case where $F_{sc} = 1$, this simply says that $G_s = J_c$, i.e., the radiant intensity incident on the sample came from the cavity walls since the sample does not see itself. Substituting Eqs. (10) and (11) into (12) results in (3) and (4), as desired.

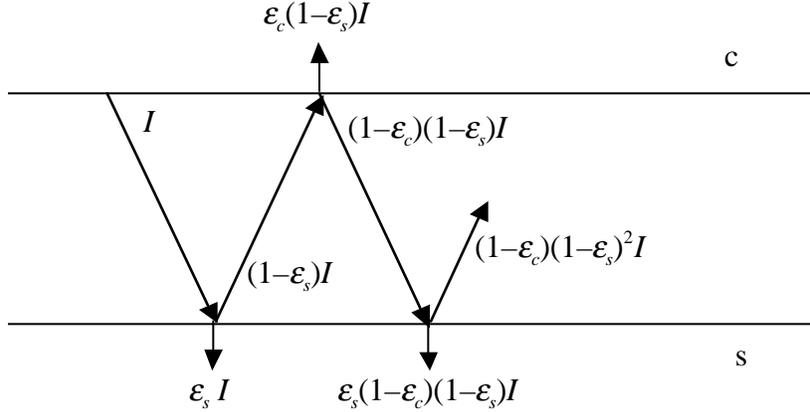
It is interesting to consider another important geometry: two infinite planes. For convenience, let's continue to denote the two surfaces by the subscripts "s" and "c" although we no longer have a cavity per se. Now $A_s / A_c = 1$, $F_{sc} = 1$, and $F_{cs} = 1$, so that Eq. (5) is still satisfied, and Eq. (6) simplifies to

$$P_{load} / A = \epsilon_{eff} \sigma (T_c^4 - T_s^4) \quad (13)$$

for the heat load per unit area of the surface, where

$$\frac{1}{\epsilon_{eff}} = \frac{1}{\epsilon_s} + \frac{1}{\epsilon_c} - 1, \quad (14)$$

again reducing to Eq. (1) only if $\epsilon_c = 1$. This result is also derived in Schaum's another way, bolstering our confidence. The rate $P(c \rightarrow s)$ at which radiant energy is emitted by plane "c" and absorbed by plane "s" is given by an infinite geometric series, as can be seen from the following diagram



where $I \equiv \epsilon_c \sigma T_c^4$ and I've used the fact that the sample's absorptivity equals ϵ_s and its reflectance equals $1 - \epsilon_s$, as discussed earlier. Thus,

$$\frac{P(c \rightarrow s)}{A} = \epsilon_s I \sum_{n=0}^{\infty} (1 - \epsilon_c)^n (1 - \epsilon_s)^n = \frac{\epsilon_s \epsilon_c \sigma T_c^4}{1 - (1 - \epsilon_c)(1 - \epsilon_s)} = \epsilon_{eff} \sigma T_c^4 \quad (15)$$

and likewise,

$$\frac{P(s \rightarrow c)}{A} = \epsilon_{eff} \sigma T_s^4. \quad (16)$$

Subtracting Eqs. (15) and (16) gives the net heat load as Eq. (13). What is instructive about this derivation, in addition to its obvious analog to the derivation of the Airy function for Fabry-Perot interference, is that it clearly indicates the error in Hecht's deduction of Eq. (7). Equation (15) can be rewritten as

$$P(c \rightarrow s) = \frac{\alpha_s A_s \cdot \epsilon_c \sigma T_c^4}{1 - (1 - \epsilon_c)(1 - \epsilon_s)} \quad (17)$$

from which we see that the sample's absorption is not $\alpha_s A_s \cdot \epsilon_c \sigma T_c^4$ unless $\epsilon_c = 1$, in which case Eq. (7) is trivially consistent with (2). It is also worth noting that the sample's total emission is not $\epsilon_s A_s \sigma T_s^4$; Eq. (16) says that we must replace ϵ_s by ϵ_{eff} .

Let's end this discussion with a consideration of two practical matters. First, consider Kirchhoff's law, Eq. (2), in a bit more detail. As is well-known, this equation can be derived by inserting a sample into a constant-temperature cavity (say by running water coils through its walls) whose internal surface we now realize must be a blackbody (say by coating it with carbon black) and waiting for the sample to thermally equilibrate with the cavity walls. It is also well-known that the equality of Eq. (2) holds not only for the total emissivity and absorptivity but also for the spectral values,

$$\alpha(\lambda) = \epsilon(\lambda) \quad (18)$$

i.e., the emission and absorption of the sample are equal wavelength by wavelength. This can be

seen by imagining that an interference filter with a very narrow passband is interposed between the sample and the cavity walls. To avoid distorting the blackbody radiation filling the cavity, we imagine the filter to be ideal, meaning it perfectly reflects all wavelengths except the passband, which it perfectly transmits; it thus absorbs nothing and consequently emits nothing either. It is now clear that the sample could not remain in equilibrium with the black cavity unless it absorbs and emits equally at the pass wavelength. This could also be seen from a modern viewpoint by taking into account the fact that photons of different wavelength carry different amounts of energy. Less well-known is the fact that equality also holds for the directional emissivity and absorptivity and not just the usual hemispherical values,

$$\alpha(\theta, \phi) = \varepsilon(\theta, \phi) \quad (19)$$

i.e., a sample emits as much thermal radiation into a particular direction as it absorbs from that same direction. The reasoning is similar to that used to obtain Eq. (18). This time we interpose a perfect mirror between the sample and the black cavity walls, and then punch a small hole in it. (In order to fix the direction for the entire sample and to prevent escape of emitted rays by multiple reflections, the geometry should consist of a small laminar sample at the center of a large spherical mirror.) An important consequence of Eq. (19), recalling once again that the absorptivity of an opaque sample equals unity minus the reflectance, is that the thermal radiation emitted by an object is polarized (as the complement of the Fresnel equations). Another way to see this result is to consider the emitting centers to be slightly below the sample surface so that the rays escape only if they do not reflect off this surface on their way out.

A second practical matter is the relaxation of a cooled sample back to the ambient temperature after the laser is shut off. Since the heat capacity of the sample is $C \equiv dQ / dT_s$,

$$dQ = P_{load} dt = CdT_s = -Cd(\Delta T) \quad (20)$$

where $\Delta T \equiv T_c - T_s$. Assuming ΔT is small (compared to the constant ambient temperature T_c), Eq. (3) can be expanded to first order to obtain

$$P_{load} \cong \frac{4\sigma T_c^3}{R} \Delta T \Rightarrow \frac{dt}{\tau} = -\frac{d(\Delta T)}{\Delta T} \quad (21)$$

from Eq. (20), where $\tau \equiv RC / 4\sigma T_c^3$ is the time constant. This can be integrated to obtain

$$\Delta T(t) = \Delta T(0)e^{-t/\tau}. \quad (22)$$

An exponential with the same time constant similarly describes the approach to the final cooling temperature (found by balancing the heat load against the cooling power) if the laser is abruptly turned on. Now observe that if $\varepsilon_c \rightarrow 0$ then $R \rightarrow \infty$ from Eq. (4) and hence $\tau \rightarrow \infty$, so that the radiative coupling rate becomes negligible. This is of course the reason that thermos bottles, cryostats, and the like are silvered or polished to a mirrorlike finish. Although this may also seem like an ideal situation for laser cooling (using tiny windows to permit the laser to enter and exit the chamber), it is in fact disastrous because it means that the fluorescence (which is carrying away heat from the sample) will be repeatedly reflected back to the sample! It is just such repeated reflections that make laser cooling difficult in GaAs, because its high index of refraction tends to prevent the fluorescence from escaping.