Small-Argument Expansion of a Polynomial in a Denominator—C.E. Mungan, Spring 2009

Suppose we want to expand the reciprocal of a polynomial in x for small x. There are many ways to do that, of which I present four methods here. To be specific, let's suppose the polynomial is $a + bx + cx^2$ where a, b, and c are all nonzero. (There are simpler techniques one can use if there are fewer nonzero terms. On the other hand, the methods I present here can be generalized if there are more than 3 nonzero terms or if different powers of x are present.) The goal is to represent the reciprocal of that polynomial in the form of the power series $a_0 + a_1 x + a_2 x^2 + \cdots$ provided that x is small.

To begin, we may as well divide a out of our polynomial, so that we are trying to expand

$$f(x) = a^{-1} \frac{1}{1 + Ax + Bx^2} \tag{1}$$

where $A \equiv b/a$ and $B \equiv c/a$.[1] In stating that x is small, what I mean is that both bx and cx^2 are smaller in magnitude than a, or in other words that $|x|$ is much less than the smaller of $|A|^{-1}$ and $|B|^{-1/2}$, say at least 10 times smaller.

The first method is brute long-hand division,

$$\begin{array}{r|l} & 1 - Ax + (A^2 - B)x^2 + \cdots \\ \hline 1 + Ax + Bx^2 & 1 \\ & 1 + Ax + Bx^2 \\ \hline & -Ax - Bx^2 \\ & -Ax - A^2x^2 - ABx^3 \\ \hline & (A^2 - B)x^2 + \cdots \end{array} \tag{2}$$

so that

$$f(x) = \boxed{\frac{1}{a} - \frac{b}{a^2}x + \left(\frac{b^2}{a^3} - \frac{c}{a^2}\right)x^2 + \cdots}\,. \tag{3}$$

A second method is to collect the two small terms in x together and expand in a geometric series,

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[1] One could eliminate a second coefficient by defining $X \equiv Ax$ and $\alpha \equiv B/A^2$ so that one seeks the reciprocal of $1 + X + \alpha X^2$. But that does not make the math much easier and it requires an extra step to rewrite the final answer back in terms of x.

$$\begin{aligned}\left[1+\left(Ax+Bx^2\right)\right]^{-1} &= 1-\left(Ax+Bx^2\right)+\left(Ax+Bx^2\right)^2-\cdots \\ &= 1-Ax+\left(A^2-B\right)x^2+\cdots\end{aligned} \tag{4}$$

in agreement with Eq. (2).

A third method is to factor out the first two terms and then expand in two different geometric series,

$$\begin{aligned}\left[(1+Ax)+Bx^2\right]^{-1} &= [1+Ax]^{-1}\left[1+Bx^2(1+Ax)^{-1}\right]^{-1} \\ &= \left[1-Ax+(Ax)^2-\cdots\right]\left[1+Bx^2(1-Ax+\cdots)\right]^{-1} \\ &= \left[1-Ax+A^2x^2-\cdots\right]\left[1+Bx^2-\cdots\right]^{-1} \\ &= \left[1-Ax+A^2x^2-\cdots\right]\left[1-Bx^2+\cdots\right] \\ &= 1-Ax+\left(A^2-B\right)x^2+\cdots\end{aligned} \tag{5}$$

which again agrees with the previous methods.

Finally, one can directly apply the standard formula for a Maclaurin series,

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\cdots \tag{6}$$

to again get the same result, as is left for the reader to verify.