Rectangular Stack of Cubes—C.E. Mungan, Summer 2021

You have N sugar cubes. You stack them into the shape of a cuboid (i.e., a rectangular prism) because you intend to exactly fill a rectangular box with the stack (which is how sugar cubes are typically packaged). If you can see three faces of the stack, how many cubes can you see? How many cubes are you not seeing?

Let the three edges of the stack have lengths x, y, and z (each of which must be positive integers) and denote the stack by [x,y,z]. Then $N = xyz$. Consequently x, y, and z must be factors of N. There is always at least one possible stack, namely [1,1,N]. This arrangement is the sole possibility if N is either prime or equal to 1. Call it the (unique) 1-1 stack, so named because at least two edges have length 1.

For a general [x,y,z] stack, the number of cubes you cannot see is $I = (x-1)(y-1)(z-1)$. For example, a [2,2,14] stack has 13 invisible cubes. (If your goal is to hide I cubes, you can always do so by making a [2,2,I+1] stack using $N = 4I + 4$ sugar lumps. Therefore you can hide 0,1,2,… cubes by making a stack out of 4,8,12,… lumps. It will turn out that $4I + 4$ is the *maximum* number of lumps required to hide I cubes. We can hide I cubes with fewer lumps if and only if I is a composite number.)

If N is neither unity nor prime (i.e, it is a composite number) then find all prime factors of N including repetitions. For example, for 84 we have {2,2,3,7}. Split this group into all possible distinct pairs of subgroups. All possible "1 stacks" are then obtained when each stack consists of 1 and the products of all entries in each subgroup. For 84 we thus get:

{2} & {2,3,7} => [1,2,42]
{3} & {2,2,7} => [1,3,28]
{7} & {2,2,3} => [1,7,12]
{2,2} & {3,7} => [1,4,21]
{2,3} & {2,7} => [1,6,14]

All N cubes are visible for the 1-1 stack (which is a line of cubes) and the 1 stacks (which are sheets of cubes). In other words $I = 0$ for all of them.

Finally if there are at least 3 prime factors in the group, split it into all possible distinct triples of subgroups. All remaining stacks are formed from the products of all entries in each subgroup. For 84 we thereby obtain:

{2} & {2} & {3,7} => [2,2,21]
{2} & {3} & {2,7} => [2,3,14]
{2} & {2,3} & {7} => [2,6,7]
{3} & {2,2} & {7} => [3,4,7]

These "0 stacks" all have some invisible cubes. There is exactly one of each kind of stack if N is the cube of a prime number, such as $N = 27$ for which the 1-1 stack is [1,1,27], the 1 stack is [1,3,9], and the 0 stack is [3,3,3] with $(N^{1/3} - 1)^3 = 2^3 = 8$ invisible and thus $27 - 8 = 19$ visible cubes.

A recent Math Contest puzzle asked about possible stacks for which 231 cubes are invisible. Since 231 factors into two 1's and primes as $I = 1 \times 1 \times 3 \times 7 \times 11$, then we must choose x and y from among these factors plus unity, i.e., from among {2,2,4,8,12}, which leaves $z = 231(x-1)^{-1}(y-1)^{-1} + 1$. So here are all the possibilities:

x	***y***	***z***	***N***
2	2	232	928
2	4	78	624
2	8	34	544
2	12	22	528
4	8	12	384

As previously claimed, the maximum number of lumps required to hide 231 cubes is $4(I+1)=928$. But we can use as little as 384 lumps to hide 231 cubes. We minimize the number of lumps required by making *x*, *y*, and *z* as nearly equal to each other as possible; if we could make them equal then $x = I^{1/3}+1$ which is approximately 7.14, in which case we would only need approximately 363.35 lumps.

For all stacks in this paper, I have arranged [*x*,*y*,*z*] such that $x \le y \le z$ to make it easier to avoid duplicates.