Remainder One Less—C.E. Mungan, Spring 2021

Problem: Given a list of positive integers $\vec{L} = \{L_1, L_2, L_3, \ldots\}$ find a formula for the smallest positive integer N such that if N is divided by any L_i the remainder is $L_i - 1$.

Example: Suppose $\vec{L} = \{6,9,6,8\}$ which emphasizes that there can be repetitions in the list. Then $N = 71$ which checks as

$$71 \div 6 = 11 \text{ rem } 5 \text{ and } 5 = 6 - 1$$
$$71 \div 8 = 8 \text{ rem } 7 \text{ and } 7 = 8 - 1$$
$$71 \div 9 = 7 \text{ rem } 8 \text{ and } 8 = 9 - 1.$$

Solution: Find the shortest list of prime factors $\vec{P} = \{P_1, P_2, P_3, \ldots\}$ which includes those of every L_i. Then

$$N = \prod P_i - 1.$$

Here the product is called the "least common multiple" of the list $\vec{L}$. In the example, 6 factors into $\{2,3\}$. Next, 9 factors into $\{3,3\}$ and so we extend our list of prime factors to $\{2,3,3\}$ to make it minimally cover both 6 and 9. Finally, 8 factors into $\{2,2,2\}$ and we end up with $\vec{P} = \{2,2,2,3,3\}$ so that

$$N = 2 \times 2 \times 2 \times 3 \times 3 - 1 = 71.$$

We can see why this works because 8 (say) divides into the first term on the right-hand side and thus the remainder is –1 which is equivalent (modulo 8) to 7. While the integer

$$M = \prod L_i - 1$$

would likewise have a remainder of $L_i - 1$ when divided by L_i, it is in general larger than N. We need to multiply together the shortest list of prime factors.

Another Example: Suppose $\vec{L} = \{1,2,3,4,5,6,7,8,9,10\}$. First notice that 1 always divides any integer with a remainder of 0 and so we can discard 1 from this list, and not worry about whether the number 1 is to be considered prime or not. Then $\vec{P} = \{2,2,2,3,3,5,7\} \Rightarrow N = 2519$.