Repeating Decimals—C.E. Mungan, Summer 2015

Prove that any repeating decimal fraction can be expressed in rational form by dividing the repetend by that number all of whose digits are 9 and having as many digits as the repetend has.

For example $0.\overline{756}$ has repetend 756. Thus it equals 756/999 = 28/37.

Suppose the repetend has the three digits abc. (The proof is similar for any other number of digits.) Then

$$0.\text{abcabcabc}\ldots = \text{abc}\left(\frac{1}{1000}+\frac{1}{1000^2}+\frac{1}{1000^3}+\cdots\right).$$

If we define $x = 1/1000$, this result becomes

$$\text{abc}\left(1+x+x^2+\cdots\right)x = \text{abc}\frac{x}{1-x} = \text{abc}\frac{1000x}{1000-1000x} = \text{abc}\frac{1}{1000-1} = \frac{\text{abc}}{999}$$

as desired.