A Representation for any Prime Number—C.E. Mungan, Summer 2019

Prove that any prime number greater than or equal to 5 can be written in the form $\sqrt{1+24n}$ where n is some positive integer.

Proof: Let the prime number be $p \geq 5$. For $n = 1$, we generate $p = 5$ and so the proposed formula starts correctly. Now consider $p^2 - 1 = (p+1)(p-1)$. Since p must be odd, then both $p+1$ and $p-1$ must be even. Thus let $(p-1)/2$ be the integer k, in which case $(p+1)/2 = k+1$. Either way, it follows that $p = 2k+1$. But since p is prime and larger than 3, $2k+1$ cannot be divisible by 3. Hence either $2k$ or $2k+2$ must be divisible by 3, which means that either k or $k+1$ must be divisible by 3. But in addition, either k or $k+1$ must be divisible by 2. Therefore $k(k+1)$ must be divisible by 6. Now observe that

$$\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right) = k(k+1)$$

which implies that $(p^2-1)/4$ is divisible by 6. We conclude that $p^2 - 1$ is divisible by 24. Call the divisor n, so that $p^2 - 1 = 24n$. QED

Comment: The formula $p(n) = \sqrt{1+24n}$ is not useful as a means to generate primes by substituting in random integer values n for two reasons. First, some integer values of n (such as 3) generate irrational numbers rather than primes. Second and more problematically, some other integer values of n (such as 26) generate integers p that are *not* prime. The fact that any prime number can be written as $\sqrt{1+24n}$ does *not* imply that any $\sqrt{1+24n}$ is a prime number!

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