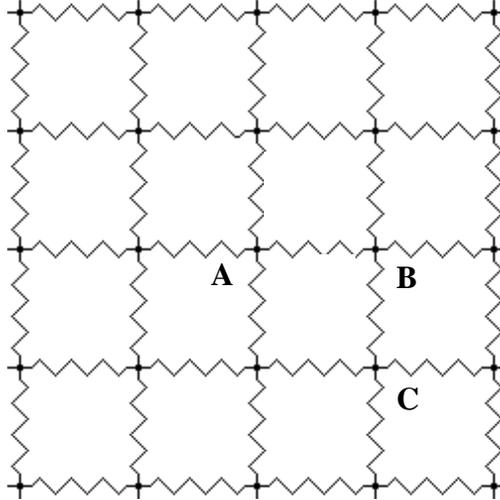


Infinite Square Lattice of Resistors—C.E. Mungan, Fall 1999

Problem: Find the equivalent resistance R_{AB} between two adjacent junctions in an infinite square lattice of $1\text{-}\Omega$ resistors.



Solution: Thanks to Robert Blumenthal for the following idea. Assume that infinity is at zero potential. This is equivalent to imagining there is a grounded conducting ring of infinite radius centered on AB. This is a sink or source of current as needed. Now suppose you have a battery of terminal potential ξ . Imagine connecting it to the above circuit in two different ways.

In the unprimed configuration, connect the positive terminal of the battery to junction A and the negative terminal to ground. A current I will consequently flow into junction A and out to the ring at infinity, subdividing into finer and finer currents as it encounters successive junctions. In particular, a current of $I/4$ flows through AB. The voltages at A and B are ξ and say V , respectively.

Next, in the primed configuration, connect the negative terminal of the battery to junction B and the positive terminal to ground. The same current I as above will consequently flow out of the ring at infinity in a myriad of fine currents which merge at successive junctions as they move in toward AB, finally becoming I flowing into the battery from junction B. Once again, we see that a current of $I/4$ flows through resistor AB. The voltages at A and B are, by symmetry, $-V$ and $-\xi$, respectively.

Now superpose the unprimed and primed configurations, by adding potentials at every junction and currents through every resistor. At very large distances from AB, the myriad of fine currents through the resistors are equal in magnitude but opposite in direction in the two configurations. Hence, in the superposed circuit, there are no currents at infinity. The net current through AB is $I/2$, and the currents through all other possible routes from A to B through the rest of the lattice must sum to $I/2$, since there is a net current of I in from the outside world at A and out at B. The superposed voltages at A and B are $V_A = \xi - V$ and $V_B = -\xi + V$, respectively. But we now have two different ways of writing the potential difference between A and B,

$$V_{AB} = IR_{AB} \quad \text{and} \quad V_{AB} = \left(\frac{1}{2}I\right)(1\ \Omega).$$

The first equation comes from the fact that we can reduce our entire lattice to an equivalent resistor R_{AB} through which the external current of I must flow. The second equation comes from the fact that in the superposed circuit, a current of $I/2$ flows through a $1\text{-}\Omega$ resistor. Comparing the right-hand sides of these two equations immediately gives the final result,

$$\boxed{R_{AB} = \frac{1}{2} \Omega.}$$

Comments: The equivalent resistance between any pair of junctions other than nearest neighbors cannot be found using the above method. For example, it turns out that $R_{AC} = 2/\pi \Omega$, which we're certainly not going to get without summing up an infinite series! In the unprimed configuration, for instance, when the $I/4$ current through AB hits junction B, it is *not* the case that one-third of this amount, or $I/12$, flows through BC.

It is also worth noting that V_{AB} is not equal to 2ξ , as Robert Blumenthal supposed. Instead $V_{AB} = 2(\xi - V)$. If V were zero, then the equivalent resistance R_∞ between say junction A and the ring at infinity would be $1/4 \Omega$. In fact, however, R_∞ is infinite! Initially, this is troubling as it implies that the battery used above must have an infinite emf ξ if a finite current I is to result in the unprimed and primed configurations. I have thus obtained R_{AB} above by subtracting two infinities (ξ and V)! So instead of putting the grounded ring at infinity, place it at some very large distance d from AB. Then in the superposed circuit, there will remain some minute currents at the ring. But we can make these distant currents as small as we like by making d large enough and hence can render them negligible.

Evidently, any one path to infinity has infinite resistance, but there are infinitely many parallel paths out to infinity, so it is not obvious whether a given infinite configuration has R_∞ finite or not. In fact, a 1D infinite ladder has R_∞ diverging linearly to infinity, a 2D lattice has R_∞ diverging logarithmically to infinity, and a 3D lattice has R_∞ finite.