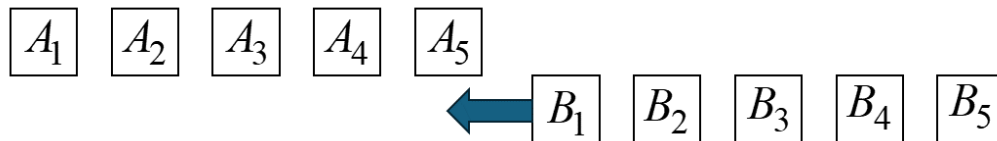


## Reversible Heat Exchange Between Two Objects at Different Temperatures—C.E. Mungan, Summer 2026

Rudnyi [1] cites a Russian thermodynamics text published in 2019 showing how two objects whose temperatures differ by a noninfinitesimal amount can be made to reversibly exchange heat by bringing them into thermal contact with each other in a clever manner. Bob Siddon and I developed a general formula for the intermediate and final temperatures of the two objects during this process. We subsequently discovered that similar formulas were previously published [2] by Mishchenko and Pshenichka (M&P) who note that the idea was presented in a Russian science book dating back significantly earlier. (Its third edition was already published in 1976.)

Suppose we have two equal quantities of the same fluid (say water as proposed by M&P, or even simpler, a monatomic ideal gas), one at 100°C and the other at 0°C. (It is much easier to use fluids rather than the solid bars suggested by Rudnyi.) If we simply brought them into contact, allowed them to equilibrate, and then separated them again, they would each end up at 50°C, which would irreversibly generate a substantial amount of entropy. We can reduce the entropy that is created by instead sliding the edge of one system laterally along the edge of the other system so that different portions of each object progressively come into thermal contact. To maximize the effect, insert adiabatic barriers between equally divided portions of each object. (This desire to insert dividing barriers explains why it is easier to use a fluid than a solid for the objects.) Even simpler, as suggested by M&P, is to pour each of the water samples into  $N$  individual beakers (where  $N$  is a positive integer). Each beaker contains  $1/N$  of the original amount of one of the water samples. The beakers are massless containers that permit heat exchange only when brought into contact with another beaker. (In other words, they prevent any other thermal loss to the surroundings by radiation or conduction.)

For example, suppose  $N = 5$ . We equally divide the hot water at 100°C among five beakers. Their temperatures are indicated by  $A_1$  through  $A_5$ , initially all equal to 100°C, but after the contacts described below, their final temperatures will each be different and indicated by adding primes to their symbols. Likewise, the cold water at 0°C is equally divided among five beakers and their temperatures are indicated initially by  $B_1$  through  $B_5$ , and finally with primes. We line up each of the set of beakers in order as follows.



We now successively bring beaker  $B_1$  into thermal contact with beakers  $A_5$ ,  $A_4$ , and so on down to  $A_1$ , permitting them to fully thermalize with each other during each contact. After contact between beakers  $A_5$  and  $B_1$ , their temperatures will both equal their average value given by

$$A'_5 = \frac{A_5 + B_1}{2} = \frac{A_5}{2} \tag{1}$$

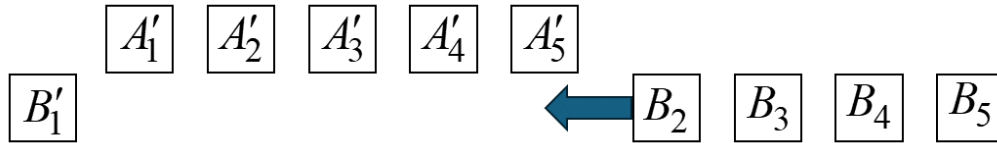
where I substituted 0 for  $B_1$  but will defer substituting 100 for the  $A$  values until later. Continuing, given that  $A'_5$  is now the temperature of beaker  $B_1$ , when it is next brought into contact with beaker  $A_4$  their temperatures will both equal the average value of

$$A'_4 = \frac{A_4 + A'_5}{2} = \frac{A_4}{2} + \frac{A_5}{4} \quad (2)$$

using Eq. (1) in the last step. This trend continues as beaker  $B_1$  is successively brought into contact with the three remaining upper beakers, and so its final temperature after passing beaker  $A_1$  is

$$B'_1 = \frac{1}{2} A_1 + \frac{1}{4} A_2 + \frac{1}{8} A_3 + \frac{1}{16} A_4 + \frac{1}{32} A_5. \quad (3)$$

The arrangement of beakers is now as shown below.



Bring beakers  $B_2$  and  $A'_5$  into contact so that both of their temperatures become

$$A''_5 = \frac{A'_5 + B_2}{2} = \frac{A'_5}{2} \quad (4)$$

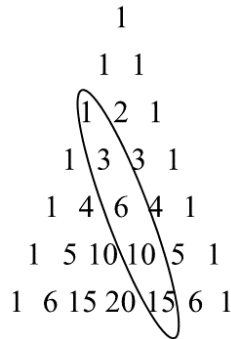
and after beaker  $B_2$  runs across the whole upper line of beakers it ends up at temperature

$$\begin{aligned} B'_2 &= \frac{1}{2} A'_1 + \frac{1}{4} A'_2 + \frac{1}{8} A'_3 + \frac{1}{16} A'_4 + \frac{1}{32} A'_5 \\ &= \frac{1}{4} A_1 + \frac{2}{8} A_2 + \frac{3}{16} A_3 + \frac{4}{32} A_4 + \frac{5}{64} A_5 \end{aligned} \quad (5)$$

after substituting Eqs. (1), (2), and likewise for the other  $A'$  values. Continuing in this manner, the final temperature of beaker  $B_3$  after it runs the gamut is

$$B'_3 = \frac{1}{8} A_1 + \frac{3}{16} A_2 + \frac{6}{32} A_3 + \frac{10}{64} A_4 + \frac{15}{128} A_5. \quad (6)$$

The numerators of the final sums of fractions in Eqs. (3), (5), and (6) are the diagonals of Pascal's triangle shown below. For example, the circled diagonal gives the numerators in Eq. (6).



Given that these triangle numbers are the binomial coefficients  $C$ , we can thus write in general

$$B'_i = \sum_{j=1}^5 \frac{A_j}{2^{i+j-1}} {}_{i+j-2}C_{i-1}. \quad (7)$$

For example, the term in this sum for  $i = 3$  and  $j = 4$  is

$$\frac{A_4}{2^6} {}_5C_2 = \frac{10}{64} A_4 \quad (8)$$

which is indeed the fourth term on the right-hand side of Eq. (6). The last step consists in pouring all the lower beakers back into one container, so that their temperatures all average together to give the final temperature of one reunited lower object. We generalize Eq. (7) to  $N$  instead of merely 5 beakers and we equate all of the  $A$  temperatures to 100 to get

$$T_{\text{lower}} = \frac{100}{N} \sum_{i=1}^N \sum_{j=1}^N \frac{{}_{i+j-2}C_{i-1}}{2^{i+j-1}} = 100 \left( 1 - \frac{2^N C_N}{2^{2N}} \right). \quad (9)$$

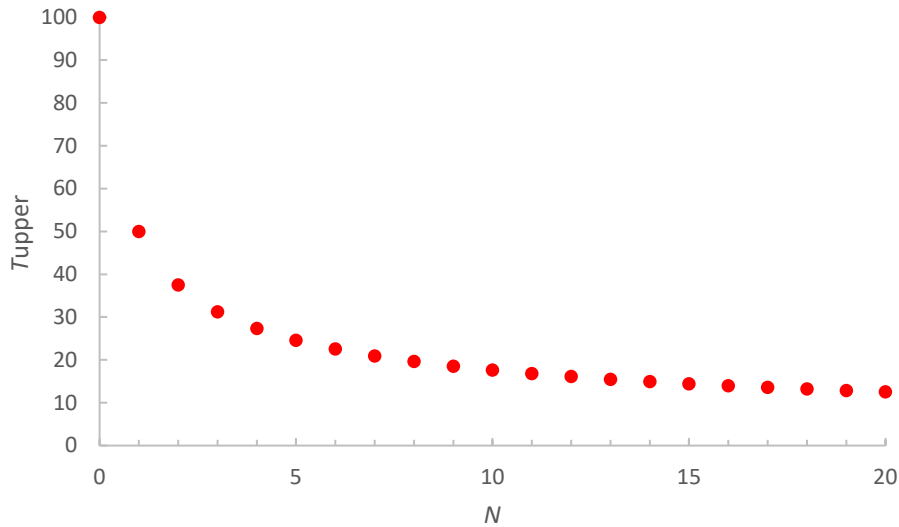
A derivation of this final form of the answer is in the appendix of M&P and can also be obtained by typing “sum C(i+j-2,i-1)/2^(i+j-1), {j=1 to N}, {i=1 to N}” into AI in the Google search bar. Likewise pour all of the upper beakers into one container, so that its final temperature will be

$$T_{\text{upper}} = 100 \frac{2^N C_N}{2^{2N}} \quad (10)$$

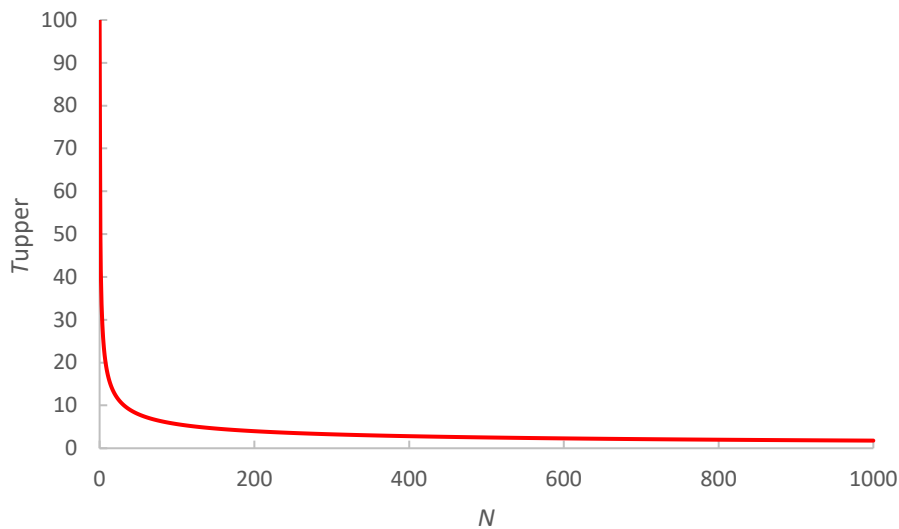
since the average temperature of the two final objects must be 50. (At any point in time, say after beaker  $B_2$  has passed beaker  $A'_4$  but before it has reached beaker  $A'_3$  in the second diagram above, if we poured every beaker into one giant container, the temperature of that mixture would be 50°C by conservation of thermal energy for calorimetry.) We can rewrite Eq. (10) as

$$\begin{aligned} T_{\text{upper}} &= 100 \frac{(2N)!}{[2^N N!]^2} = 100 \frac{(2N)(2N-1)(2N-2)(2N-3)\cdots(3)(2)(1)}{[2(N)2(N-1)\cdots 2(1)]^2} \\ &= 100 \frac{(2N-1)(2N-3)\cdots(3)(1)}{2(N)2(N-1)2(N-2)\cdots 2(2)2(1)} = 100 \frac{(2N-1)(2N-3)\cdots(3)(1)}{(2N)(2N-2)(2N-4)\cdots(4)(2)} \\ &= 100 \frac{(2N-1)!!}{(2N)!!} = \boxed{100 \frac{1 \ 3 \ 5 \ 7 \ \dots}{2 \ 4 \ 6 \ 8}}. \end{aligned} \quad (11)$$

As a simple check, for  $N = 1$  we get  $T_{\text{upper}} = 50^\circ\text{C}$  as expected for just bringing two beakers, one at  $0^\circ\text{C}$  and one at  $100^\circ\text{C}$ , into contact. Let's define the case of not bringing the two beakers into contact at all (so that  $T_{\text{upper}} = 100^\circ\text{C}$  does not change) as  $N = 0$  for graphing purposes. Then the top of the next page shows an Excel plot for  $N$  running from 0 to 20.



We can easily continue this graph out to  $N = 1000$ , in which case the curve becomes essentially continuous, as shown next.



Clearly in the limit that  $N \rightarrow \infty$  we get  $T_{\text{upper}} = 0^\circ\text{C}$  and thus  $T_{\text{lower}} = 100^\circ\text{C}$ . That limiting value can be formally proven by applying the Stirling approximation to the first form in Eq. (11) to get

$$T_{\text{upper}} \approx \frac{100}{\sqrt{\pi N}}. \quad (12)$$

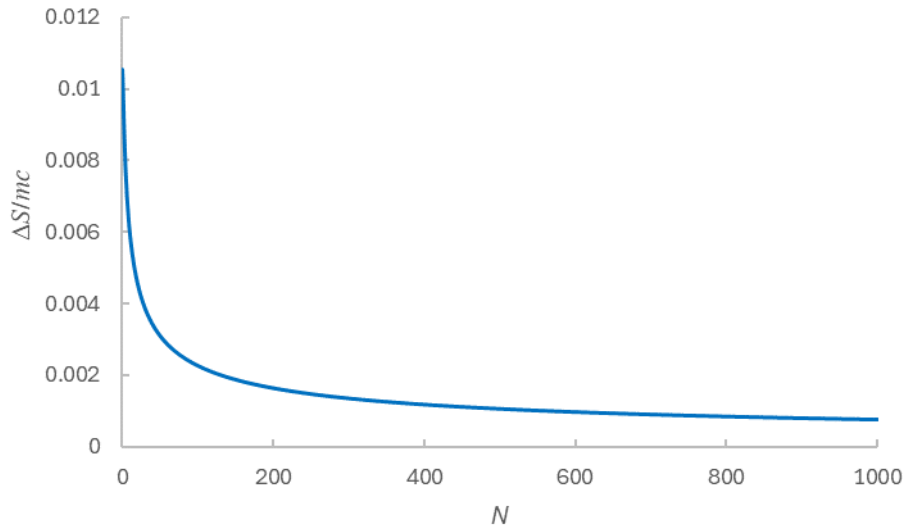
However, in this infinite limit, the heat transfer must be reversible. Namely, to reverse it, simply run the infinite set of lower beakers back rightward over the line of upper beakers! A reversible process occurs when heat is transferred in infinitesimal increments  $dQ$ . Normally that occurs when an object of finite mass  $m$  changes temperature by an infinitesimal increment  $dT$  such that

$dQ = mcdT$  where  $c$  is the specific heat of the object. Here it occurs because an object of infinitesimal mass  $dm$  changes temperature by a finite increment  $\Delta T$  such that  $dQ = dmc\Delta T$ .

The entropy change of the system can be explicitly calculated. The upper sample of water starts out at 373 K and ends up at 273 K +  $T_{\text{upper}}$  (with  $T_{\text{upper}}$  in °C) whereas the lower sample starts out at 273 K and ends up at 373 K -  $T_{\text{upper}}$ . If  $m$  is the mass of either sample and  $c$  is the specific heat of water, then the total normalized entropy change is

$$\frac{\Delta S}{mc} = \ln \left[ \frac{273 + T_{\text{upper}}}{373} \right] + \ln \left[ \frac{373 - T_{\text{upper}}}{273} \right] \quad (13)$$

as plotted in the following graph. As both this graph and Eq. (13) show, the total entropy change is zero in the reversible limit of  $T_{\text{upper}} = 0$ .



[1] E. Rudnyi, “Reversible processes in classical thermodynamics,” online preprint at <https://www.preprints.org/manuscript/202510.1821>

[2] E.G. Mishchenko and P.F. Pschenichka, “Reversible temperature exchange upon thermal contact,” *Am. J. Phys.* **85**, 23–29 (Jan. 2017). There are minor typos in Eqs. (24) and (25), but (26) remains correct.