Integral Representation of the Riemann Zeta Function—C.E. Mungan, Fall 2001

Prove that

$$\zeta(s) \equiv \sum_{k=1}^{\infty} k^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \tag{1}$$

where $s > 1$.

Tim Royappa communicated to me the following wonderfully compact solution. The trick is to use the Laplace transform

$$\mathcal{L}(x^{s-1}) \equiv \int_0^{\infty} e^{-kx} x^{s-1} dx = \Gamma(s) k^{-s}. \tag{2}$$

We prove Eq. (2) as follows. Let $u \equiv kx > 0$. Then

$$\mathcal{L}(x^{s-1}) = \int_0^{\infty} e^{-u} \left(\frac{u}{k}\right)^{s-1} d\left(\frac{u}{k}\right) = \frac{1}{k^s} \int_0^{\infty} e^{-u} u^{s-1} du \equiv k^{-s} \Gamma(s) \tag{3}$$

as required.

Now from Eq. (2) we have

$$\begin{aligned} \sum_{k=1}^{\infty} k^{-s} &= \frac{1}{\Gamma(s)} \int_0^{\infty} \left(\sum_{k=1}^{\infty} e^{-kx} \right) x^{s-1} dx = \frac{1}{\Gamma(s)} \int_0^{\infty} \left\{ \left(\sum_{k=0}^{\infty} e^{-kx} \right) - 1 \right\} x^{s-1} dx \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} \left(\frac{1}{1 - e^{-x}} - 1 \right) x^{s-1} dx = \frac{1}{\Gamma(s)} \int_0^{\infty} \left(\frac{e^{-x}}{1 - e^{-x}} \right) x^{s-1} dx \end{aligned} \tag{4}$$

which simplifies to Eq. (1).