Riemann Zeta of 4 as Needed for Stefan-Boltzmann Law—C.E. Mungan, Spring 2010

The purpose of this note is to derive the fact that

$$\zeta(4) \equiv \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} . \tag{1}$$

The derivation proceeds in two steps.

First we find the Fourier series (FS) for a unit antisymmetric triangular wave $f(t)$. Since the wave has zero average value, there is no constant term in the FS. Also, since the wave is antisymmetric, the FS is a pure sine series and it suffices to consider half a cycle from $t = 0$ to $t = T/2$. Finally, since the wave is symmetric about $t = T/4$, the FS only contains odd harmonics and we can restrict attention to a quarter cycle from $t = 0$ to $t = T/4$,

$$f(t) = \sum_{m=1,3,5,\ldots} b_m \sin\left(m \frac{2\pi}{T} t \right) \tag{2}$$

where the coefficients are 4 times the integral over one quarter cycle,

$$b_m = 4 \frac{2}{T} \int_0^{T/4} f(t) \sin\left(m \frac{2\pi}{T} t \right) dt = \frac{8}{T} \int_0^{T/4} \frac{t}{T/4} \sin\left(m \frac{2\pi}{T} t \right) dt \tag{3}$$

where $f(t)$ is defined to equal 1 when $t = T/4$. We integrate by parts to get

$$\begin{aligned} b_m &= \frac{32}{T^2} \left[-t \frac{T}{2\pi m} \cos\left(m \frac{2\pi}{T} t \right) \Bigg|_0^{T/4} + \frac{T}{2\pi m} \int_0^{T/4} \cos\left(m \frac{2\pi}{T} t \right) dt \right] \\ &= \frac{32}{T^2} \left[0 + \left(\frac{T}{2\pi m} \right)^2 \sin\left(m \frac{2\pi}{T} t \right) \Bigg|_0^{T/4} \right] = (-1)^{(m-1)/2} \frac{8}{\pi^2 m^2} \end{aligned} . \tag{4}$$

Inserting this result in Eq. (2) and switching dummy summation index using $m = 2n+1$ gives

$$f(t) = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\left[(2n+1) \frac{2\pi}{T} t \right]. \tag{5}$$

The second step is to apply Parseval's relation in the form pertinent to the present FS,

$$\left\langle f^2(t) \right\rangle = \frac{1}{2} \sum_{n=0}^{\infty} b_n^2 \tag{6}$$

where the angle brackets denote the average of the wave over a period. In our case, that is equivalent to finding the average of the parabola $y = x^2$ over the range from $x = 0$ to $x = 1$,

$$\langle y \rangle = \int_0^1 x^2 \, dx = \frac{1}{3} \tag{7}$$

since our triangular wave amounts to a linear ramp from zero up to unity over a quarter cycle. Substituting this result into the left-hand side of Eq. (6) and the coefficients from Eq. (5) into the right-hand side, we have

$$\frac{1}{3}=\frac{1}{2}\frac{64}{\pi^4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} \quad \Rightarrow \quad \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}=\frac{\pi^4}{96}\,. \tag{8}$$

Finally we equate this sum over only odd integers to the desired sum over all integers minus the sum over only even integers,

$$\frac{\pi^4}{96}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}=\sum_{n=1}^{\infty}\frac{1}{n^4}-\sum_{n=1}^{\infty}\frac{1}{(2n)^4}=\frac{15}{16}\sum_{n=1}^{\infty}\frac{1}{n^4} \tag{9}$$

where the last equality came from factoring $1/2^4=1/16$ out of the sum over even integers. Rearranging Eq. (9) now gives Eq. (1).