Kinetic Energy of a Rigid Body—C.E. Mungan, Fall 2000

Introductory textbooks such as Serway are not very clear about whether the kinetic energy of a rigid body is to be calculated by summing together translational and rotational terms or by using the translational or rotational formulae alone. The answer is that all three of these choices give the same answer. In this note, I briefly review how to properly apply these three options and demonstrate that they do in fact give the same answer.

<u>Choice 1</u>: $K_{tot} = K_{TR} = \sum \frac{1}{2} m_i \upsilon_i^2$

This is the definition of the kinetic energy of an extended body or system and so it is always correct. It is useful in problems where the system is composed of a finite number of parts, each of which are executing pure translations. The whole purpose of introducing rotational kinetic energy is to simplify the process of performing the above summation in the case of a continuous, rotating object. Therefore, choices 2 or 3 are usually easier for such objects, assuming their moments of inertia are known or can be easily calculated.

<u>Choice 2</u>: $K_{tot} = K_{TR} + K_R = \frac{1}{2} M \upsilon_{CM}^2 + \frac{1}{2} I_{CM} \omega^2$

The idea here is to decompose the motion into a pure translation of the center of mass plus a pure rotation of the object about its center of mass, so that

$$\boldsymbol{\upsilon}_i = \boldsymbol{\upsilon}_{CM} + \boldsymbol{\omega} \times \mathbf{r}_i$$

where $\mathbf{r}_i$ is the position of the i^{th} particle in the system relative to the center of mass. Thus, the total kinetic energy is by definition

$$\sum \tfrac{1}{2} m_i \upsilon_i^2 = \tfrac{1}{2}\left(\sum m_i\right)\upsilon_{CM}^2 + \tfrac{1}{2}\sum m_i (\boldsymbol{\omega}\times\mathbf{r}_i)\cdot(\boldsymbol{\omega}\times\mathbf{r}_i) + \sum m_i \boldsymbol{\upsilon}_{CM}\cdot(\boldsymbol{\omega}\times\mathbf{r}_i).$$

Consider each of the three terms on the right-hand side of this expression. The quantity in the parentheses of the first term is just the total mass M of the body. The third term can be rewritten as

$$\boldsymbol{\upsilon}_{CM}\cdot\left(\boldsymbol{\omega}\times\sum m_i \mathbf{r}_i\right) = \boldsymbol{\upsilon}_{CM}\cdot\left(\boldsymbol{\omega}\times M\mathbf{r}_{CM}\right).$$

But $\mathbf{r}_{CM} = 0$ since the origin has been chosen to be at the center of mass. Finally the second term can be simplified using well-known expressions for triple products. First we permute the “cross” and “dot” in the left-hand triple scalar product to get

$$\tfrac{1}{2}\sum m_i \boldsymbol{\omega}\cdot[\mathbf{r}_i\times(\boldsymbol{\omega}\times\mathbf{r}_i)].$$

Next we apply the BAC–CAB rule to the triple vector product in square brackets to obtain

$$\tfrac{1}{2}\sum m_i \boldsymbol{\omega}\cdot[\boldsymbol{\omega} r_i^2 - \mathbf{r}_i(\boldsymbol{\omega}\cdot\mathbf{r}_i)] = \tfrac{1}{2}\left(\sum m_i r_i^2\right)\omega^2 - \tfrac{1}{2}\sum m_i(\boldsymbol{\omega}\cdot\mathbf{r}_i)^2 .$$

Let’s choose the z-axis to lie in the direction of $\boldsymbol{\omega}$. Consequently this term becomes

$$\tfrac{1}{2}\left[\sum m_i\left(x_i^2 + y_i^2 + z_i^2\right)\right]\omega^2 - \tfrac{1}{2}\sum m_i(\omega z_i)^2 = \tfrac{1}{2}\left(\sum m_i \rho_i^2\right)\omega^2$$

where ρ_i is the cylindrical radial coordinate. But the quantity in parentheses in this last expression is just the moment of inertia about the center of mass. Putting together what we have found for the three terms gives the desired form for the total kinetic energy.

Choice 3: $K_{tot} = K_R = \frac{1}{2} I_0 \omega^2$

We can always find some origin 0 about which the system is instantaneously undergoing pure rotations. Hence the kinetic energy is solely rotational provided we calculate the moment of inertia about that origin. This can most easily be done using the parallel axis theorem,

$$I_0 = I_{CM} + Ml^2$$

where l is the distance from the axis 0 to the center of mass. Substituting this into our proposed form for K_R and noting that $\omega l = \upsilon_{CM}$ immediately shows it to be equivalent to choice 2. Note that the angular speeds of the system about 0 and about the center of mass are equal.

A careful student will practice her understanding of the above results by explicitly calculating the kinetic energy different ways for some example problems and showing that she gets the same results:

A. Serway P10.54 on page 300 deals with two isolated astronauts of equal mass M tethered by a rope of length d and revolving about their center of mass with tangential speed υ. Show that the kinetic energy is $M\upsilon^2$, both by treating the astronauts as two point masses and applying choice 1, and by treating the system as a single system executing pure rotations about the center of mass and using choice 3.

B. Serway E10.13 on page 287 considers a uniform rod of length L and mass M rotating on a frictionless pin through one end. Show that the kinetic energy at any instant is given by $ML^2\omega^2 / 6$, both by considering pure rotation about the end according to choice 3, and by decomposing the motion into translation of the center of mass plus rotation about the center of mass using choice 2.