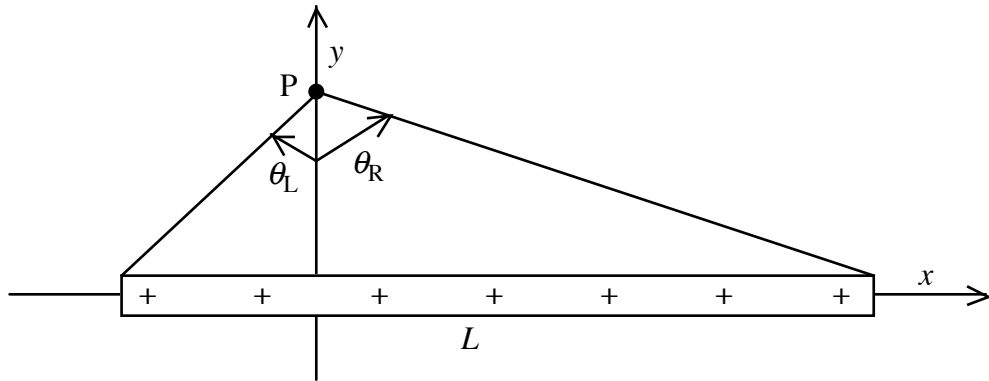


Electric Field of a Uniformly Charged Straight Rod—C.E. Mungan, Spring 2014

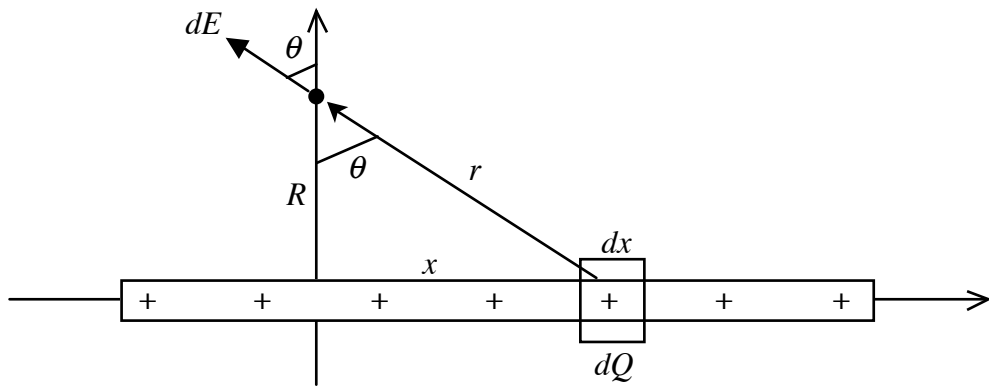
It is relatively simple to find a general expression for the electric field of a uniform rod at any arbitrary point in space. The result includes the case of the field on the axis of the rod beyond one of its ends, and the case of an infinitely long rod. The general answer is most conveniently expressed in terms of the linear charge density λ ; for a finite rod of length L and total charge Q , that charge density is equal to Q/L . To begin with, I will assume that the field point is a perpendicular distance R away from the axis of the rod, and later I will discuss what happens if one is interested in the case of $R = 0$ beyond one of the ends.

Start by positioning the x axis along the rod, and choose the y axis so that it passes through the point P at which we wish to determine the field, as sketched below.



For specificity, I have assumed that Q is positive, but the results apply to either sign of charge. The right end of the rod is located at angle θ_R measured counter-clockwise from the y axis. Using the same sign convention, the angle θ_L of the left end of the rod indicated above happens to be negative, but it would be positive if P were located to the left of the left end of the rod.

We find the components of the electric field by chopping the rod into bits of charge dQ located a distance x at angle θ away from the field point P .



The x component of the electric field is seen from this diagram to be

$$E_x = -\int \frac{kdQ}{r^2} \sin \theta \tag{1}$$

and the y component is

$$E_y = \int \frac{k dQ}{r^2} \cos \theta . \quad (2)$$

Insert $dQ = \lambda dx$ into both of these equations. Next comes the key step that makes this problem much easier than at first appears. Note that

$$\frac{x}{R} = \tan \theta \Rightarrow \boxed{\frac{dx}{r^2} = \frac{R d\theta}{r^2 \cos^2 \theta} = \frac{d\theta}{R}} . \quad (3)$$

This formula is so simple that it is worth deriving another way, as is done in Appendix A. Its use makes the two preceding integrals trivial,

$$E_x = -\frac{k\lambda}{R} \int_{\theta_L}^{\theta_R} \sin \theta d\theta = \boxed{\frac{k\lambda}{R} (\cos \theta_R - \cos \theta_L)} \quad (4)$$

and similarly

$$E_y = \boxed{\frac{k\lambda}{R} (\sin \theta_R - \sin \theta_L)} . \quad (5)$$

These are the final general results. It is useful to specialize them to various interesting cases.

Case 1: On the perpendicular bisector of a finite rod, $\theta_R \equiv +\theta$ say and $\theta_L = -\theta$. Then Eq. (4) implies that $E_x = 0$, as agrees with the left-right symmetry of the problem, while Eq. (5) becomes

$$E_y = \frac{2k\lambda}{R} \sin \theta = \frac{2k\lambda}{R} \frac{L/2}{\sqrt{R^2 + (L/2)^2}} = \boxed{\frac{kQ}{R\sqrt{R^2 + L^2/4}}} . \quad (6)$$

This result correctly becomes the usual point-charge field kQ/R^2 if $R \gg L$.

Case 2: For an infinitely long rod, $\theta_R = +90^\circ$ and $\theta_L = -90^\circ$. Again Eq. (4) gives $E_x = 0$, while Eq. (5) correctly reduces to the familiar Gauss's law result

$$E_y = \boxed{\frac{2k\lambda}{R}} \quad (7)$$

which can also be obtained from the third term in Eq. (6) by taking the limit as $L \rightarrow \infty$.

Case 3: Above the right end of a finite rod, $\theta_R = 0$ and $\theta_L \equiv -\theta$ say. Then Eq. (4) gives

$$E_x = \frac{k\lambda}{R} (1 - \cos \theta) = \boxed{\frac{k\lambda}{R} \left(1 - \frac{R}{\sqrt{R^2 + L^2}} \right)} \quad (8)$$

and Eq. (5) becomes

$$E_y = \frac{k\lambda}{R} \sin\theta = \frac{k\lambda}{R} \frac{L}{\sqrt{R^2 + L^2}}. \quad (9)$$

Case 4: Above the right end of a semi-infinite rod, $\theta_R = 0$ and $\theta_L = -90^\circ$, so that Eqs. (4) and (5) give

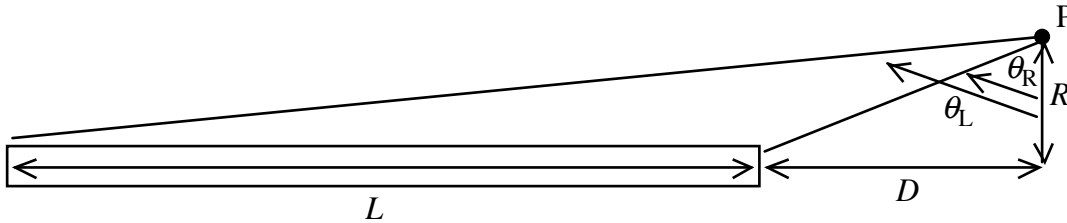
$$E_x = \frac{k\lambda}{R} \quad (10)$$

and

$$E_y = \frac{k\lambda}{R}, \quad (11)$$

respectively. These two equations can also be obtained from Eqs. (8) and (9), respectively, by taking the limit as $L \rightarrow \infty$. They imply that the field above the end of a semi-infinite rod always points 45° away from the axis of the rod, regardless of how near or far the field point is from the rod! Also notice that Eq. (11) has half of the value of Eq. (7), which makes sense because if we butt together the ends of two equally-charged semi-infinite rods, we get an infinite rod. (Meanwhile the x components of the two rods are equal and opposite, so that they cancel.)

Case 5: On the axis of a finite rod at a distance D from the right end, the diagram initially becomes the following, where eventually we want to take the limit as $R \rightarrow 0$.



We see from this diagram that

$$\sin\theta_R = \frac{-D}{\sqrt{R^2 + D^2}} \rightarrow -1 \text{ as } R \rightarrow 0 \quad (12)$$

and similarly

$$\sin\theta_L = \frac{-(D+L)}{\sqrt{R^2 + (D+L)^2}} \rightarrow -1 \text{ as } R \rightarrow 0. \quad (13)$$

Thus Eq. (5) correctly implies that $E_y = 0$, provided one ignores the fact that the prefactor of $k\lambda/R$ in Eq. (5) diverges as $R \rightarrow 0$. (For a more satisfactory treatment of the limit, see Appendix B.) The diagram likewise shows that

$$\frac{k\lambda}{R} \cos\theta_R = \frac{k\lambda}{\sqrt{R^2 + D^2}} \rightarrow \frac{k\lambda}{D} \text{ as } R \rightarrow 0 \quad (14)$$

and

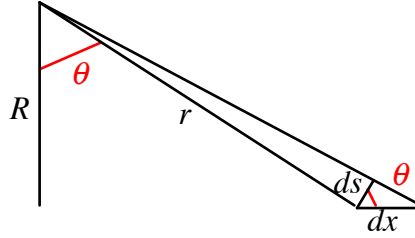
$$\frac{k\lambda}{R} \cos\theta_L = \frac{k\lambda}{\sqrt{R^2 + (D+L)^2}} \rightarrow \frac{k\lambda}{D+L} \text{ as } R \rightarrow 0. \quad (15)$$

Substituting these two results in Eq. (4) finally gives

$$E_x = \frac{k\lambda}{D} - \frac{k\lambda}{D+L} = \boxed{\frac{kQ}{D(D+L)}}. \quad (16)$$

As for case 1, this result correctly becomes the usual point-charge field kQ/D^2 if $D \gg L$.

Appendix A: Alternate derivation of Eq. (3)



From this diagram we see that, using the definition of angle in radians,

$$d\theta = \frac{ds}{r} = \frac{dx}{r} \cos\theta = \frac{dx}{r} \frac{R}{r} \quad (A.1)$$

which rearranges into Eq. (3).

Appendix B: E_y on the axis of a finite rod

Substituting the middle terms in Eqs. (12) and (13) into (5), one obtains

$$\frac{-E_y}{k\lambda} = \frac{1}{R} \left[\frac{D}{\sqrt{R^2 + D^2}} - \frac{X}{\sqrt{R^2 + X^2}} \right] = \frac{1}{R} \left[\left(1 + \frac{R^2}{D^2} \right)^{-1/2} - \left(1 + \frac{R^2}{X^2} \right)^{-1/2} \right] \quad (B.1)$$

where $X \equiv D + L$. For small R , expand both square roots using the binomial expansion,

$$\frac{-E_y}{k\lambda} \rightarrow \frac{1}{R} \left[\left(1 - \frac{R^2}{2D^2} \right) - \left(1 - \frac{R^2}{2X^2} \right) \right] = \frac{R}{2} \left(\frac{1}{X^2} - \frac{1}{D^2} \right) \rightarrow 0 \quad (B.2)$$

in the limit as $R \rightarrow 0$, as we wished to show. In contrast, if we considered a point on the rod, then θ_L and θ_R would have opposite signs, so that the sign between the two terms in parentheses in the last term of Eq. (B.1) would be plus rather than minus. In that case, Eq. (B.2) shows that E_y would diverge, just as it does for an infinite rod as $R \rightarrow 0$ according to Eq. (7).