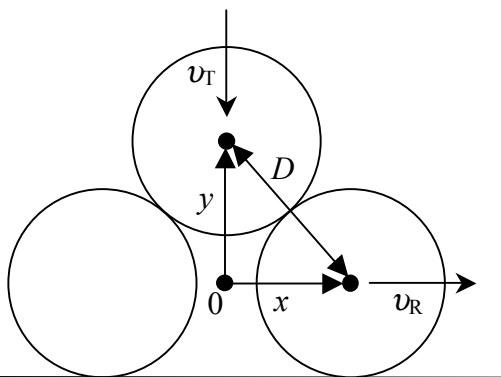


Rolling Stack of Cans—C.E. Mungan, Summer 2006

Three homogeneous cylindrical cans are stacked horizontally on a flat table in the shape of an equilateral triangle, as seen from the side. The cans are released from rest. Ignoring friction, what is the speed of each can at the moment that the top can strikes the table?

Choose a coordinate system with the origin 0 directly below the center of the top can and along the line joining the centers of the two lower cans. Let x be the distance from the origin to the center of the right-hand lower can (which has speed $v_R = dx / dt$) and let y be the distance from the origin to the center of the top can (which has speed $v_T = -dy / dt$). An end view of the cans at some arbitrary instant in time while the cans remain in contact with each other is sketched below, where D is the diameter of a can. Initially $x_i = D / 2$, $y_i = h = \sqrt{3}D / 2$, and $v_{Ri} = v_{Ti} = 0$.



As long as the top can makes contact with the right-hand can, Pythagoras' theorem says

$$x^2 + y^2 = D^2. \quad (1)$$

Differentiating this equation with respect to time leads to

$$xv_R = yv_T. \quad (2)$$

But conservation of mechanical energy implies that the kinetic energy gained by the three cans equals the potential energy lost by the top can,

$$2g(h - y) = v_T^2 + 2v_R^2. \quad (3)$$

Substitute v_T from Eq. (2) and x from Eq. (1) to obtain

$$\frac{v_R^2}{2g} = \frac{(h - y)y^2}{D^2 + y^2}. \quad (4)$$

The cans lose contact when v_R and hence the right-hand side of this expression is a maximum, which occurs when y equals some critical value y_c ,

$$\frac{d}{dy}(hy^2 - y^3)(D^2 + y^2)^{-1} = 0 \Rightarrow y_c^3 + 3D^2y_c - 2hD^2 = 0. \quad (5)$$

Rewrite this critical value in dimensionless form as $Y \equiv y_c / D$,

$$Y^3 + 3Y - \sqrt{3} = 0. \quad (6)$$

The only real solution of this reduced cubic equation (as you can check using the cubic formula or by plotting the left-hand side) is

$$Y = \left(\frac{\sqrt{3} + \sqrt{7}}{2} \right)^{1/3} - \left(\frac{2}{\sqrt{3} + \sqrt{7}} \right)^{1/3} \cong 0.528222, \quad (7)$$

which properly lies between zero and $\sqrt{3}/2$ in value. Since the right-hand can travels at constant speed after the top can loses contact with it, its final speed is

$$v_{Rf}^2 = \frac{(\sqrt{3} - 2Y)Y^2}{1 + Y^2} gD \Rightarrow v_{Rf} \cong 0.383906\sqrt{gD} \quad (8)$$

from Eq. (4) with $y = y_c$, and therefore

$$v_{Tf}^2 = \frac{4Y^3 - \sqrt{3}Y^2 + \sqrt{3}}{1 + Y^2} gD \Rightarrow v_{Tf} \cong 1.198867\sqrt{gD} \quad (9)$$

from Eq. (3) with $y = 0$ and $v_R = v_{Rf}$.