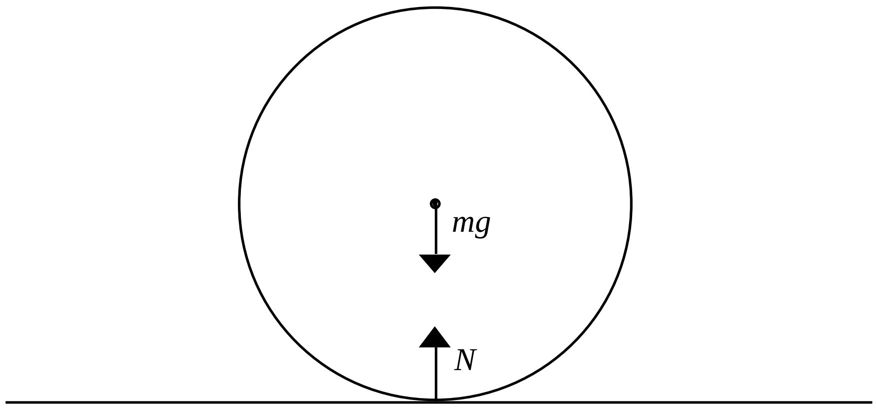
Rolling Friction of a Free Wheel—C.E. Mungan, Spring 2001

Define a free wheel to be an isolated disk (i.e., having no axle) which therefore can have neither a drive nor a brake mechanism and for which air resistance is negligible. It is well known that if such a wheel were rigidly circular and rolled on a rigidly flat surface (which combination we might refer to as an "ideal wheel") that its free-body diagram would consist of only two forces, its weight and the normal force due to the road, as sketched below.

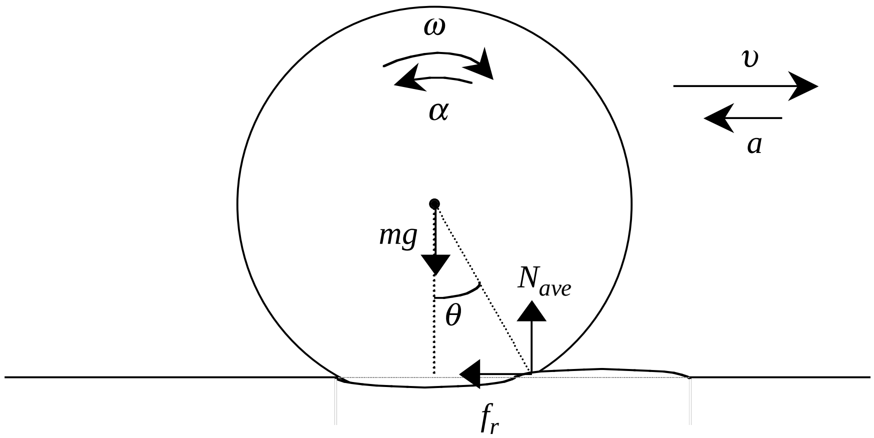


The frictional force must be zero because it cannot logically point either forward or backward. If it were in the forward direction, it would translationally accelerate the wheel; if it pointed rearward, it would rotationally accelerate the wheel. To see this another way, suppose the wheel rolled from pavement onto a slick ice patch. It would continue to both translate and spin (i.e., roll without slipping) without interruption, demonstrating that friction plays no role.

Clearly the above scenario cannot hold for a real wheel, which gradually slows down and comes to rest on flat ground. In general, a wheel deforms from a perfectly circular figure—there must be a finite, flattened area of contact which when multiplied by the average internal and sidewall pressure (if we think of a tire) couples the weight and the normal force. (Increasing the weight primarily changes the contact area not the pressure. You may wish to verify this using a tire gauge before and after filling your car with passengers.) In addition, a wheel digs somewhat into a road, so the pavement is not perfectly flat. (Loaded 18-wheelers, for example, are particularly harsh on asphalt highways.) Regardless of whether the asphericity of the wheel or the nonplanarity of the road dominates, the effect is the same. The effective contact force on the wheel shifts to an average position located forward of the center line and is inclined toward the rear. (The leading portion of the wheel and surface are compressed more than the trailing portion, resulting in a larger Hookean response. Energy is lost as heat by two mechanisms: slippage of the leading and trailing edges of the wheel—which unlike the bottom-most point of the wheel are not at rest relative to the pavement—resulting in kinetic friction, and the fact that neither the wheel nor road are perfectly elastic so that there is hysteresis associated with the compression and relaxation cycles.) We can resolve this contact force into a vertical and a horizontal component. I will refer to the vertical component as the average normal force N_{ave} and the horizontal portion as the rolling frictional force f_r. We can now define a coefficient of rolling friction in the usual manner as

$$\mu_r \equiv f_r / N_{ave}. \qquad (1)$$

(Note that many authors choose to multiply the right-hand side of this expression by the radius R of the wheel. In the model I will develop, such a scaling factor does not naturally arise.) Young and Freedman cite typical values for μ_r of 0.01–0.02 for rubber tires on concrete and 0.002–0.003 for steel wheels on steel rails, thereby explaining the greater potential efficiency of locomotive travel. Hence a free-body diagram for a real free wheel rolling to the right is something like the following.



The frictional force provides the translational deceleration,

$$f_r \equiv \mu_r N_{ave} = ma. \tag{2}$$

Note from the vertical force balance that $N_{ave} = mg$, and hence μ_r can be directly measured as the fractional acceleration relative to gravity, i.e., $\mu_r = a/g$. On the other hand, the torque about the center due to the normal force is larger than that due to the frictional force and thus results in the rotational deceleration,

$$\begin{aligned} \tau = N_{ave}R\sin\theta - f_r R\cos\theta &= maR\left(\frac{\sin\theta}{\mu_r} - \cos\theta\right), \\ &= I\alpha = \gamma maR \end{aligned} \tag{3}$$

using Eq. (2) twice in the second step. I have assumed the wheel rolls without slipping to relate a to α, and have introduced the dimensionless coefficient $\gamma \equiv I/mR^2$. Assuming θ is small, we therefore obtain

$$\theta \cong \mu_r(1+\gamma). \tag{4}$$

This relationship could alternatively have been derived by equating the torque about the bottom-most point of the wheel (about which pure rotations occur), which is approximately $N_{ave}R\sin\theta$, to α multiplied by $I + mR^2$ using the parallel-axis theorem. For example, if $\gamma = 0.5$ (as for a uniform disk) and $\mu_r = 0.015$, then $\theta = 1.3^\circ$. This gives a measure of the deformation of the wheel and road.

Note that if a wheel is driven by a motor providing a torque τ_{motor} about the axle, or retarded by a brake producing a torque τ_{brake}, that it is not necessary to take into account the preceding non-ideal deformations. In the driven case, we can assume there is a static frictional force f_s at the bottom-most point which is in the forward direction (since the wheel has a tendency to slip backward as it presses against the pavement) to provide the translational acceleration a, while $\tau_{motor} = f_s R + I\alpha$ is in the direction of rotation of the wheel to generate the rotational acceleration α. Since there is a maximum magnitude for the static friction (depending on the coefficient μ_s), the wheel will not slip provided one does not accelerate it too hard. In the case of the braked wheel, we reverse the directions of both f_s and the applied torque.