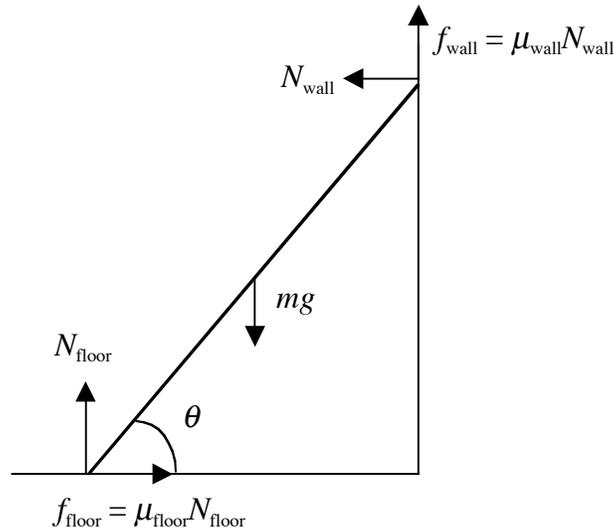


Two Problems of Rotational Equilibrium—C.E. Mungan, Summer 2000

1. An example problem in most algebra and calculus based textbooks is the following. A ladder of uniform mass density leans against a wall, making an angle θ with the floor. What is the minimum value of θ if the ladder is not to slip?

Without exception, every introductory textbook on my shelf adds the stipulation that the wall is to be considered smooth, so that its coefficient of static friction is $\mu_{\text{wall}} = 0$. Marion and Hornyak comment, “If friction exists between the top of the ladder and the wall, the system becomes indeterminate.” Sears and Zemansky similarly say, “Such a problem is statically indeterminate. The difficulty is that it’s no longer adequate to treat the body as being perfectly rigid.” (To be fair, it’s not clear if S&Z are referring to the more general problem of finding the forces when the ladder is at an arbitrary angle or specifically to the case of the ladder on the verge of slipping. No such confusion exists in M&H however.) In fact, the problem is perfectly soluble even when the coefficients μ_{floor} and μ_{wall} are both nonzero.

Since the ladder is on the verge of slipping, the frictional forces at both the wall and floor must be at their maximum values: if they were not, the ladder could not slip. A free-body diagram is thus as follows.



Balancing the horizontal forces gives

$$N_{\text{floor}} = N_{\text{wall}} / \mu_{\text{floor}}, \quad (1)$$

while the vertical force balance is

$$N_{\text{floor}} = mg - \mu_{\text{wall}} N_{\text{wall}}. \quad (2)$$

Equating the right-hand sides of Eqs. (1) and (2) and rearranging gives a solution for N_{wall} . Finally, the torque balance about the point of contact between the ladder and floor implies

$$mg \frac{1}{2} L \cos \theta = N_{\text{wall}} L \sin \theta + \mu_{\text{wall}} N_{\text{wall}} L \cos \theta. \quad (3)$$

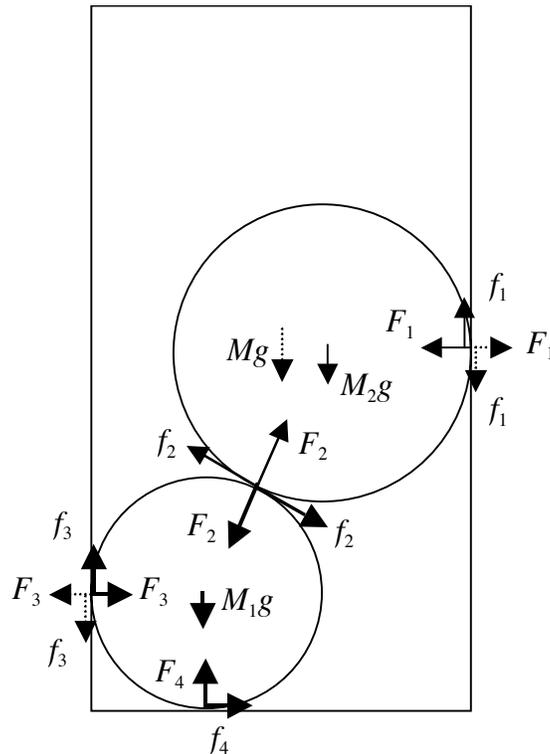
Dividing this expression through by $mgL\cos\theta$ and substituting our solution for N_{wall} gives the answer,¹

$$\theta = \tan^{-1}\left(\frac{1}{2\mu_{\text{floor}}} - \frac{\mu_{\text{wall}}}{2}\right). \quad (4)$$

This reduces to the usual textbook solution when $\mu_{\text{wall}} = 0$. In particular, the ladder can stand only at 90° if the floor is smooth, $\mu_{\text{floor}} = 0$. Notice now however that the ladder is stable at any angle if the geometric mean of μ_{floor} and μ_{wall} is greater than or equal to unity.

2. Demo 03–26 “Toppling Cylinders” in the Video Encyclopedia of Physics Demonstrations consists of a hollow right cylinder having a top cap. When the cap is removed, the cylinder falls over. The secret to this surprising behavior is that there are two balls inside the cylinder stacked on top of each other whose diameters are less than that of the cylinder. When the cap is removed, the cylinder’s mass is sufficiently reduced so that it topples.

Suppose that the cylinder has mass M and radius R , the lower ball has mass M_1 and radius R_1 , and the upper ball has mass M_2 and radius R_2 . We can construct a force diagram as follows, where forces on the upper ball are thin lines, those on the lower ball are thick lines, and those on the cylinder are dashed lines and with F indicating normal forces and f denoting frictional forces.



¹Footnote added Dec 2006: See K.S. Mendelson, “Statics of a ladder leaning against a rough wall,” Am. J. Phys. **63**, 148 (1995) for a discussion of the fact that slipping begins at the wall and then occurs at the floor due to the elastic loading of the ladder.

Here I have used Newton's third law to write all action-reaction pairs of forces. By considering the torque balance about the center of the upper ball, we see that $f_1 = f_2$. The only way these can always be equal regardless of the nature of the surfaces is if both are zero. This makes sense because the ball has no tendency either to slip or to spin even if it is greased. Similarly we see that $f_3 = f_4$ and so again both must be zero. Next, suppose that a line joining the centers of the two balls together makes an angle θ with respect to the vertical. By considering the horizontal force balance on each ball we see that

$$F_2 \sin\theta = F_1 = F_3, \quad (5)$$

while the vertical force balance on the upper ball gives

$$F_2 \cos\theta = M_2 g. \quad (6)$$

Dividing Eq. (5) by (6) gives an expression for the normal force exerted by either ball on the cylinder walls. These two normal forces constitute a couple displaced vertically from each other by a distance equal to the vertical displacement between the centers of the two balls, namely

$$y = (R_1 + R_2) \cos\theta. \quad (7)$$

The cylinder will thus topple provided its mass is less than some maximum value M_{\max} whose value is obtained by considering the torques about the lower right-hand corner of the cylinder,

$$M_{\max} g R = F_1 y. \quad (8)$$

Finally, the angle θ is obtained from the geometrical restriction that the two balls span the diameter of the cylinder,

$$(R_1 + R_2)(1 + \sin\theta) = 2R \equiv D. \quad (9)$$

Substituting our solutions for the normal force and vertical displacement into Eq. (8) and using (9) gives

$$\frac{M_{\max}}{M_2} = \frac{D - \bar{D}}{R} \quad (10)$$

where the arithmetic mean diameter of the two balls is $\bar{D} \equiv (2R_1 + 2R_2)/2$.

If the balls have equal diameters, there are two obvious limits to their diameters. On the high side, the cylinder will never topple if the diameter of the balls matches that of the cylinder. The other limit is when the balls have a diameter equal to the radius of the cylinder. In that case the two balls will lie on the bottom side by side and the cylinder again can never topple. While the former limit is evident from Eq. (10), the latter is not but is implicit in the preceding equations such as (9). Hence the limits are $1 < \bar{D}/R < 2$.

If the balls have unequal diameters, say $\Delta R \equiv |R_2 - R_1| > 0$, it is possible for the two balls to lie on the bottom even if the sum of their diameters is greater than the diameter of the cylinder. Furthermore, on the high side, the larger ball will not fit in by the time their average diameter equals that of the cylinder. The revised geometrical limits are straightforward albeit messy to

derive. The result is

$$1 + \frac{r^2}{4} < \frac{\bar{D}}{R} < 2 - r \quad (11)$$

where $r \equiv \Delta R / R$. In order for there to be any positive solution for M_{\max} at all, we require that the fractional difference in radii of the balls be no more than $r_{\max} = 2(\sqrt{2} - 1) = 0.828$.

In the graph below the final results are plotted for $r = 0$ (solid curve), 0.4 (dashed curve), and 0.8 (dash-dotted curve). The sharp corners in the plots assume rigid, smooth objects and would presumably be rounded off in a real experiment.

