Same Number of Heads—C.E. Mungan, Spring 2021

You flip a penny N times, and a nickel M times. What is the probability that you get the same number of heads for both coins?

The number of ways that you can get n heads out of N flips is $C(N,n)$. For example, there are 3 ways to get 1 head and 2 tails if you flip a penny 3 times, which is

$$C(3,1) = \frac{3!}{(3-1)!1!} = 3 . \tag{1}$$

Specifically, you could get a head on the first, second, or third flip.

Each flip can be either heads or tails. Thus there are 2^N ways to get all possible results. So the probability of getting n heads in N flips is

$$P_{n \text{ out of } N} = \frac{C(N,n)}{2^N} . \tag{2}$$

For example, the probability is 3/8 that you get two heads when you flip a penny three times.

Now let $m = \min(N,M)$ be the smaller of N and M. We can get between 0 and m heads in common between the two sets of coin flips. Since the two coins are independent, the probability of success is thus

$$p \equiv \sum_{n=0}^{m} P_{n \text{ out of } N} P_{n \text{ out of } M} = \boxed{2^{-(N+M)} \sum_{n=0}^{\min(N,M)} C(N,n) C(M,n)} . \tag{3}$$

For example, if $N = 3$ and $M = 4$ then the probability is

$$2^{-7} \sum_{n=0}^{3} C(3,n) C(4,n) = \frac{1\cdot 1 + 3\cdot 4 + 3\cdot 6 + 1\cdot 4}{2^7} = \frac{35}{128} \approx 27.3\% . \tag{4}$$

A special case is when $N = M$. Then we can use the formula

$$\sum_{n=0}^{N} C(N,n) C(N,n) = C(2N,N) \tag{5}$$

which follows from the Vandermonde identity, to find

$$p = \frac{(2N)!}{(N!2^N)^2} . \tag{6}$$

For example, if both coins are flipped three times, then

$$p = \frac{20}{64} = 31.25\% . \tag{7}$$

This is the same probability $p = 2^{-2N} C(2N,N)$ that we get $N = 3$ specific outcomes (say 2 heads and 1 tail) for half of the results of flipping $2N = 6$ coins. Namely, whatever number of heads and tails we get for the penny, we must get the same three outcomes (viz. the same number of heads and tails) for the tosses of the nickel.