Gravitational Force due to a Spherical Shell—C.E. Mungan, Spring 2014


Here is an elegant proof that a uniform spherical shell exerts a gravitational force outside it as if the shell of mass were compressed into a point at its center. In the following figure, consider the gravitational force $dF$ between a unit point mass at $P$ and differential area $dS$ on the shell of surface mass density $\sigma$. Choose the $z$ axis to pass through the center of the shell such that point $P$ lies along that axis. By symmetry, there is another patch of area (indicated in red) below the $z$ axis, which shows that the net force between $P$ and the shell must be directed along the $z$ axis.

The $z$ component of the force between $P$ and the top red element of the shell is

$$dF = \frac{G\sigma dS}{\rho^2} \cos \theta$$

where distance $\rho$ and angle $\theta$ are indicated on the next diagram. (In this diagram, the red line is an extension of the line labeled with the radius $R$ of the shell, and the blue line is perpendicular to the line labeled $\rho$.)

The relation between the area $dS$ and the solid angle $d\Omega$ in the top figure, and the angle $\alpha$ relative to the blue line in the bottom figure, is

$$d\Omega = \frac{dS}{\rho^2} \cos \alpha .$$

Substituting Eq. (2) into (1) gives
\[ dF = \frac{G\sigma d\Omega}{\cos \alpha} \cos \theta. \quad (3) \]

Now rotate the element of solid angle around the \( z \) axis to integrate out the azimuthal angle and leave only the differential of the polar angle \( \theta \),

\[ dF = \frac{G\sigma}{2\pi} \frac{\sin \theta d\theta}{\cos \alpha} \cos \theta = \frac{G\sigma \pi}{\cos \alpha} d\left(\sin^2 \theta\right). \quad (4) \]

Next apply the law of sines to the triangle in the second figure,

\[ \frac{\sin \theta}{R} = \frac{\sin(\pi - \alpha)}{r} = \frac{\sin \alpha}{r}, \quad (5) \]

so that Eq. (4) becomes

\[ dF = \frac{G\sigma \pi R^2}{r^2} \frac{d\left(\sin^2 \alpha\right)}{\cos \alpha} = \frac{G\sigma \pi R^2}{r^2} d\left(1 - \cos^2 \alpha\right) = -\frac{G\sigma 2\pi R^2}{r^2} d(\cos \alpha). \quad (6) \]

Notice that the top portion of the shell has red patches of area with \( \alpha \) varying from a minimum value of 0 to a maximum value of \( \pi / 2 \). We integrate over that range of values of \( \alpha \) and then double the answer to account for the green patches of area on the back side of the shell (relative to \( P \) in the first figure), to get

\[ F = -\frac{G\sigma 4\pi R^2}{r^2} \cos \alpha|_{\pi/2}^{0} = \frac{GM}{r^2} \quad (7) \]

where \( M \) is the mass of the shell. Equation (7) is the desired result.