Model of a Viscoelastic Solid—C.E. Mungan, Fall 2012

refs: *TPT* ***50***, *527 (Dec. 2012) & AJP* ***80***, *870 (Oct. 2012)*

Rod Cross presents experimental data for the deformation distance x and time t when a steel ball (of mass $m = 2$ kg) is dropped (from a height of few centimeters so its impact speed is 0.75 m/s) onto a cylinder of silly putty (which is 35 mm in height and 20 mm in diameter). The cylinder is on top of a piezoelectric disk to measure the impact force F during the collision. The ball is observed to bounce off as we see in the left panel of Fig. 1 below, but there is some hysteresis as evidenced in the right panel. Mechanical energy is lost during the inelastic collision, corresponding to a coefficient of restitution of 45% determined by measuring the rebound speed using a video camera.

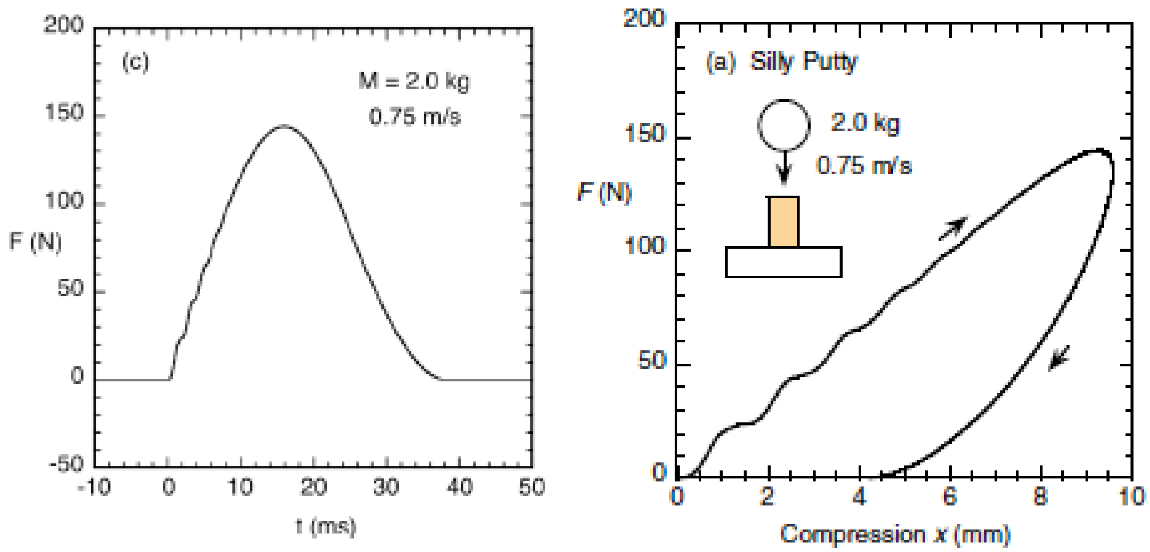

Fig. 1. The left-hand graph of F versus t is from Fig. 4 of the AJP paper, while the right-hand graph of F versus x is from Fig. 2 of the TPT paper.

Silly putty is a viscoelastic solid, so we can model it as a series combination of a Hookean spring of stiffness constant k and elongation x_s with a dashpot of drag constant b and displacement x_d, as sketched in Fig. 2.

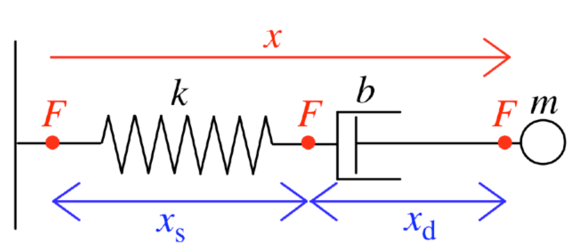

Fig. 2. A series combination of a spring and a dashpot exerting force F on a mass m.

Since the force F equals the tension which is everywhere the same along the combination (neglecting the masses of the spring and dashpot), we can write

$$F_x = -kx_\mathrm{s} = -b\dot{x}_\mathrm{d} \, . \tag{1}$$

Introducing the damping coefficient $\gamma \equiv k / 2b$ (to be contrasted with the usual expression $\gamma \equiv b / 2m$ when the spring and dashpot are in *parallel*) and natural frequency $\omega \equiv \sqrt{k / m}$ (both in rad/s), Eq. (1) implies

$$\dot{x}_\mathrm{d} = 2\gamma x_\mathrm{s} \tag{2}$$

whose time derivative is

$$\ddot{x}_\mathrm{d} = 2\gamma \dot{x}_\mathrm{s} \, . \tag{3}$$

Noting from Fig. 2 that the total compression of the silly putty is $x = x_\mathrm{s} + x_\mathrm{d}$ and that the force on the ball is $F_x = m\ddot{x}$, we conclude that

$$m\ddot{x}_\mathrm{s} + m\ddot{x}_\mathrm{d} = -kx_\mathrm{s} \tag{4}$$

using the first equality in Eq. (1). Substituting Eq. (3) into (4) gives

$$\ddot{x}_\mathrm{s} + 2\gamma \dot{x}_\mathrm{s} + \omega^2 x_\mathrm{s} = 0 \tag{5}$$

which we recognize as the standard equation of a damped undriven oscillator with solution

$$x_\mathrm{s}(t) = Ae^{-\gamma t} \sin \omega' t \tag{6}$$

assuming underdamping ($\gamma < \omega$) where the amplitude is A, the damped frequency is $\omega' = (\omega^2 - \gamma^2)^{1/2}$, and a sine solution has been chosen to match the initial conditions $x_\mathrm{s} = 0$ and $x_\mathrm{d} = 0$ at $t = 0$. We substitute this solution into Eq. (2) to find

$$x_\mathrm{d}(t) = 2\gamma A \int_0^t e^{-\gamma t} \sin \omega' t \, dt = 2\gamma A \frac{\omega' \left(1 - e^{-\gamma t} \cos \omega' t\right) - \gamma e^{-\gamma t} \sin \omega' t}{\omega'^2 + \gamma^2} \, . \tag{7}$$

Adding together Eqs. (6) and (7) gives

$$x = A \frac{\left(\omega'^2 - \gamma^2\right) e^{-\gamma t} \sin \omega' t + 2\gamma \omega' \left(1 - e^{-\gamma t} \cos \omega' t\right)}{\omega'^2 + \gamma^2} \tag{8}$$

whereas Eq. (6) substituted into the first equality in Eq. (1) gives

$$F = kAe^{-\gamma t} \sin \omega' t \tag{9}$$

in magnitude, limited to the range of times $0 \le t \le T / 2$ while the ball is in contact with the silly putty, where the period is $T \equiv 2\pi / \omega'$.

In Excel, I computed F and x versus t and plot the results in Fig. 3 after varying the parameters to obtain a good match to Fig. 1. Specifically the coefficients are $k = 20$ kN/m (so that $\omega = 100$ rad/s), $b = 600$ kg/s (so that $\gamma = 16.7$ rad/s and $\omega' = 98.6$ rad/s), and $A = 9$ mm.

The peak force is on the order of kA, the maximum value of x is on the order of A, and the collision time is $\pi / \omega' = 31.9$ ms .

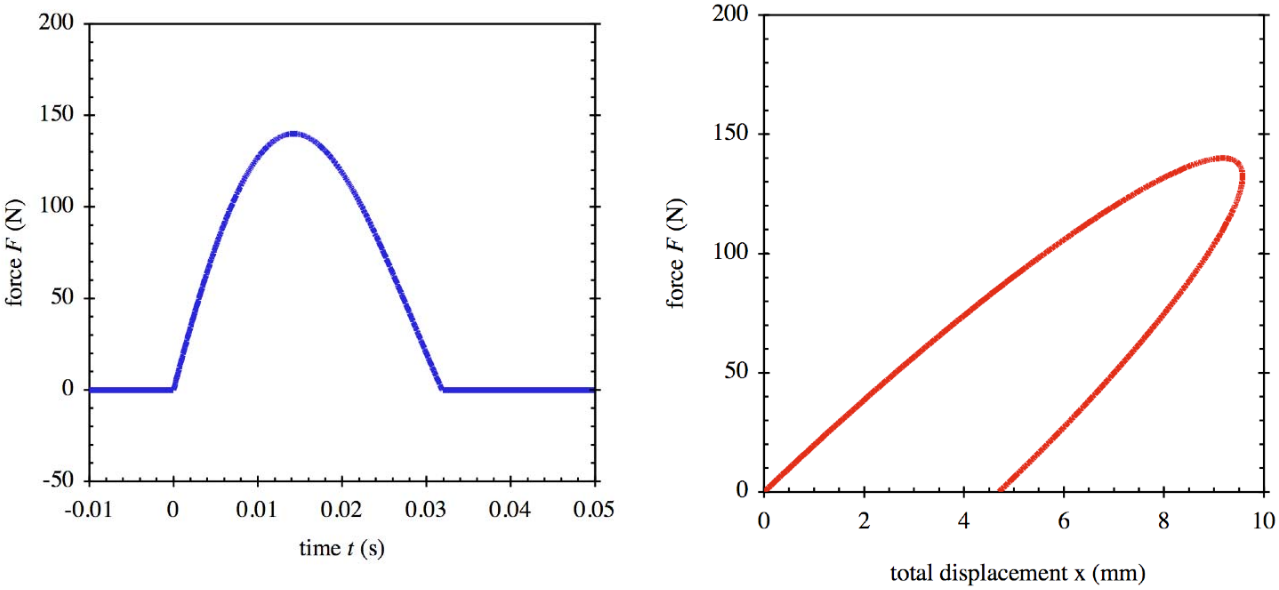

Fig. 3. Predicted force versus time or compression for the parameters given in the text.

So far the agreement between theory and experiment has been good. Less satisfactory however is the calculation of the coefficient of restitution, COR, which was experimentally found to be 45%. Integrating the area under the left-hand graph in Fig. 3 gives an impulse of $I = 2.82\ \text{N}\cdot\text{s}$. Given that the incident speed is $\upsilon_\text{i} = 0.75$ m/s , the final speed is predicted to be

$$\upsilon_\text{f} = \frac{I}{m} - \upsilon_\text{i} = 0.66\ \text{m/s} \quad \Rightarrow \quad \text{COR} = \frac{\upsilon_\text{f}}{\upsilon_\text{i}} = 88\% . \tag{10}$$

An alternative is to compute the nonconservative work W done on the silly putty, by integrating the area enclosed by the hysteretic curve in the right-hand graph in Fig. 3, to get $W = 519$ mJ . Then

$$K_\text{f} = K_\text{i} - W \quad \Rightarrow \quad \upsilon_\text{f} = 0.21\ \text{m/s} \quad \Rightarrow \quad \text{COR} = \frac{\upsilon_\text{f}}{\upsilon_\text{i}} = 28\% . \tag{11}$$

These results are very sensitive to the exact value of the incident speed; changing υ_i by a few tenths of m/s leads to values of COR that equal 45%. Note that if we turn off the dashpot by letting $b \to \infty$, the hysteresis is correctly observed to become zero so that Eq. (11) predicts COR = 100%. However, Eq. (10) does not predict a COR of 100% unless the incident speed equals the maximum speed of the oscillator, $\upsilon_{\text{max}} = A\omega = 0.9$ m/s , which again disagrees by a few tenths from the actual incident speed. It is possible that better agreement with the observed COR would be found by using a pair of spring-dashpot systems in parallel, as Rod Cross suggests in the AJP paper.