

## Solving $M$ equations in $N$ unknowns by Gaussian elimination—C.E. Mungan, Fall 2011

First form the augmented matrix and row reduce it to echelon form. The number of solutions of the system of equations can now be read off the echelon matrix starting from the bottom row and working upward. The number of solutions is as follows.

case of  $M = N$ :

1. If the determinant of the coefficient matrix  $\mathbf{A}$  is nonzero, then there is a *unique solution*. This is the case for Example 1 on page 412 of McQuarrie. (In fact,  $\det \mathbf{A} = -2$  in this example.)
2. If the determinant of  $\mathbf{A}$  is zero and the echelon matrix has one or more rows in which every entry EXCEPT the last one is zero, then there is *no solution*. This is the case for Example 3 on page 414. (Please check for yourself that the determinant of  $\mathbf{A}$  really is zero in this example!)
3. If the determinant of  $\mathbf{A}$  is zero and situation 2 does not occur, then the echelon matrix must have one or more rows in which every entry (including the last one) is zero and there are an *infinite number of solutions*. This is the case for Example 2 on page 413.

case of  $M < N$  (underdetermined):

1. If the echelon matrix has one or more rows in which every entry EXCEPT the last one is zero, then there is *no solution*. This is the case for Example 6 on page 416.
2. Otherwise there are an *infinite number of solutions*. This includes the possibility that one or more rows consist of all zeroes, as is the case for Equation (8) on page 416.

case of  $M > N$  (overdetermined):

1. If the echelon matrix has one or more rows in which every entry EXCEPT the last one is zero, then there is *no solution*. This is the case for Example 4 on page 415.
2. If situation 1 does not occur and the echelon matrix has exactly  $N$  nonzero rows in it, then there is a *unique solution*. This is the case for Equation (7) on page 414. (There are 2 nonzero rows and there are 2 unknowns.)
3. If neither situation 1 nor 2 occurs, then the echelon matrix must have fewer than  $N$  nonzero rows in it and there are an *infinite number of solutions*. This is the case for Example 5 on page 415. (There is only 1 nonzero row but there are 2 unknowns.)